THE ROYAL STATISTICAL SOCIETY

GRADUATE DIPLOMA EXAMINATION

NEW MODULAR SCHEME

introduced from the examinations in 2009

MODULE 3

SPECIMEN PAPER B

SOLUTIONS ARE CONTAINED IN A SEPARATE FILE

The time for the examination is 3 hours. The paper contains eight questions, of which candidates are to attempt **five**. Each question carries 20 marks. An indicative mark scheme is shown within the questions, by giving an outline of the marks available for each part-question. The pass mark for the paper as a whole is 50%.

The solutions should not be seen as "model answers". Rather, they have been written out in considerable detail and are intended as learning aids. For this reason, they do not carry mark schemes. Please note that in many cases there are valid alternative methods and that, in cases where discussion is called for, there may be other valid points that could be made.

While every care has been taken with the preparation of the questions and solutions, the Society will not be responsible for any errors or omissions.

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Note. In accordance with the convention used in all the Society's examination papers, the notation log denotes logarithm to base *e*. Logarithms to any other base are explicitly identified, e.g. $\log_{10}$.

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Turn over

1. Suppose that two players, A and B, are playing a simple game against each other. A sequence of plays, whose results are independent of each other, with a stake of £1 per play, continues until one or other of the players loses all his capital, i.e. is ruined. Player A starts with capital £a and Player B starts with capital £b. Let $N = a + b$, the total capital in pounds of both players combined. At each play, Player A has probability θ of winning, where $0 < \theta < 1$, and Player B has probability $1 - \theta$ of winning. At each play, £1 is transferred from the loser to the winner.

(a) Let x_i denote the probability that Player A is eventually ruined, given that currently he has capital £i remaining, where $0 \leq i \leq N$.

(i) Write down a difference equation satisfied by the x_i, together with appropriate boundary conditions.

(2)

(ii) Assuming that $\theta \neq ½$, solve the difference equation of part (i) together with its boundary conditions.

(7)

(iii) Deduce that at the start of the game the probability that Player A is eventually ruined is given by

$$\frac{\left(\frac{1-\theta}{\theta}\right)^N - \left(\frac{1-\theta}{\theta}\right)^a}{\left(\frac{1-\theta}{\theta}\right)^N - 1}.$$

(1)

(b) Let y_i denote the expected number of plays remaining until one or other of the players is ruined, given that Player A currently has capital £i remaining, where $0 \leq i \leq N$.

(i) Write down a difference equation satisfied by the y_i, together with appropriate boundary conditions.

(2)

(ii) Assuming that $\theta \neq ½$, solve the difference equation of part (b)(i) together with its boundary conditions.

(7)

(iii) Deduce that the expected number of plays at the start of the game until ruin of one or other of the players is given by

$$\frac{a}{1-2\theta} + \frac{N}{1-2\theta}\,\frac{1-\left(\frac{1-\theta}{\theta}\right)^a}{\left(\frac{1-\theta}{\theta}\right)^N - 1}.$$

(1)

Turn over

2. (i) Let Y be a discrete random variable with

$$P(Y = y) = (1-\phi)^{y-1}\phi, \quad y = 1, 2, \dots ,$$

where $0 < \phi < 1$. For all $k = 1, 2, \dots$, show that

$$P(Y = k + y \mid Y > k) = P(Y = y), \qquad y = 1, 2, \dots .$$

(4)

(ii) In a model of how a certain bank with just one teller (server) operates, the number of customers in the system at time n (including any being served) is denoted by X_n ($n = 0, 1, 2, \dots$). The service times of all customers, in whole numbers of time units, are independent (and independent of the arrivals process), and follow the distribution given in (i). Customers arrive singly, arrivals being possible only at the times 0, 1, 2, … ; the probability of an arrival at any time n is θ ($0 < \theta < 1$), independently of all other times. Use the result in (i) to justify the claim that the process $\{X_n\}$ is a Markov chain, and write down its transition probabilities.

(6)

(iii) Let $\{\pi_j\}$ ($j \geq 0$) denote the stationary distribution of this Markov chain. In the special case where $\phi = ½$ and $\theta = ¼$, show that the π_j satisfy the following equations.

$$\pi_0 = \frac{3}{2}\pi_1$$

$$\pi_1 = \frac{1}{2}\pi_0 + \frac{3}{4}\pi_2$$

$$\pi_j = \frac{1}{4}\pi_{j-1} + \frac{3}{4}\pi_{j+1} \qquad (j \geq 2)$$

Show that $\pi_0 = ½$ and obtain an explicit expression for π_j ($j \geq 1$).

(10)

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3. Consider a linear birth and death process $\{N(t)\}$ $(t \geq 0)$ with per capita birth rate λ and per capita death rate μ, which is, equivalently, the continuous time Markov chain with state space the set of all non-negative integers and instantaneous transition rates given as follows.

transition	rate	
$i \to i+1$	λi	$(i \geq 1)$
$i \to i-1$	μi	$(i \geq 1)$

Let a denote the initial population size. Define

$$p_n(t) = P(N(t) = n \mid N(0) = a) \qquad (n \geq 0)$$

and

$$G(z, t) = \sum_{n=0}^{\infty} p_n(t)\, z^n .$$

(i) Derive the forward equations for the $p_n(t)$ $(n \geq 0)$.

(6)

(ii) Deduce that $G(z, t)$ satisfies the partial differential equation

$$\partial G/\partial t = (\lambda z - \mu)(z - 1)\partial G/\partial z.$$

(5)

(iii) Let

$$m(t) = E(N(t) \mid N(0) = a) \qquad (t \geq 0).$$

By differentiating the partial differential equation of part (ii) with respect to z and then setting $z = 1$, show that $m(t)$ satisfies the differential equation

$$dm/dt = (\lambda - \mu)m(t).$$

(5)

(iv) Deduce an expression for $m(t)$ $(t \geq 0)$.

(4)

Turn over

4. Consider the simple M/M/1 model for a queue with arrival rate λ and service rate μ, but modified to be the model for a queue "with discouragement", so that a customer who arrives to find i customers ahead of him in the queue only joins the queue with probability $1/(i + 1)$. Thus the resulting continuous time Markov chain model has as its state space the set of all non-negative integers and the following instantaneous transition rates.

transition	rate	
$i \to i + 1$	$\lambda/(i + 1)$	$(i \geq 0)$
$i \to i - 1$	μ	$(i \geq 1)$

where $\lambda > 0$ and $\mu > 0$.

(i) Write down the detailed balance equations and show that a Poisson equilibrium distribution exists for all parameter values.

(5)

(ii) Show that in equilibrium the mean rate at which customers join the queue is given by $\mu(1 - e^{-\rho})$, where $\rho = \lambda/\mu$. (Note that this is not the same as the rate λ at which customers arrive at the queue, because not all the customers who arrive at the queue actually join it.)

(5)

(iii) Find an expression in terms of ρ for the equilibrium probability that a customer who arrives at random joins the queue.

(3)

(iv) Deduce the equilibrium distribution of the number of customers ahead of a customer joining the queue.

(7)

[Note. In examination questions for this module, the word "queue" refers to all units in a system, i.e. those being served as well as those still waiting to be served.]

5. Consider the ARMA(p, q) model

$$Y_t = \sum_{k=1}^{p} \phi_k Y_{t-k} + \varepsilon_t + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i} \qquad (-\infty < t < \infty),$$

where $\{\varepsilon_t\}$ is a white noise process with variance σ^2.

(i) Define the corresponding *autoregressive characteristic polynomial* $\phi(z)$ and *moving average characteristic polynomial* $\theta(z)$. Write down a compact expression of the model in terms of these characteristic polynomials and the lag operator (backward shift operator) L.

(3)

(ii) Define the *infinite moving average expression* for the process $\{Y_t\}$ and explain how the existence of such an expression is related to the stationarity of $\{Y_t\}$.

(3)

(iii) State the condition involving a characteristic polynomial that must be satisfied for $\{Y_t\}$ to be stationary.

(2)

Consider now the special case of the model where

$$Y_t = \frac{1}{2} Y_{t-1} + \varepsilon_t + \varepsilon_{t-1} + \frac{1}{4} \varepsilon_{t-2} \qquad (-\infty < t < \infty).$$

(iv) Show that the process $\{Y_t\}$ is stationary.

(2)

(v) Obtain the infinite moving average representation of $\{Y_t\}$.

(7)

(vi) Show that $\text{Var}(Y_t) = 55\sigma^2/12$.

(3)

Turn over

6. Consider modelling a time series $\{Y_t\}$ of the average monthly exchange rate of the Latvian lat against the US dollar for the 168 months that cover the years from 1994 to 2007.

(i) The following table lists the autocorrelation function (acf) for the raw data and the acf and partial autocorrelation function (pacf) for the first differences.

Lag	Raw data acf	First differences acf	First differences pacf
1	0.954	0.268	0.268
2	0.896	–0.073	–0.156
3	0.848	–0.005	0.066
4	0806	0.112	0.092
5	0.762	0.018	–0.045
6	0.717	–0.098	–0.074
7	0.682	0.017	0.074
8	0.645	0.140	0.093
9	0.601	0.073	0.014
10	0.559	0.092	0.124
11	0.515	0.110	0.057
12	0.466	–0.038	–0.117
13	0.415	–0.028	0.040
14	0.374	0.022	0.013
15	0.337	0.008	–0.034
16	0.299	–0.021	0.008
17	0.261	0.051	0.079
18	0.222	0.081	–0.011
19	0.180	–0.026	–0.076
20	0.137	–0.118	–0.071

Explain the reasons for taking first differences. Discuss what you can conclude from the above table about reasonable ARIMA models to fit to the data.

(6)

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Turn over

A computer package is used to fit an ARIMA model to the data. The following is an abbreviated version of the output.

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ARIMA Model: rate

Final Estimates of Parameters

Type        Coef  SE Coef      T      P
MA   1  -0.3434   0.0734  -4.68  0.000

Differencing: 1 regular difference
Number of observations:  Original series 168, after
differencing 167
Residuals:    SS =  0.00885666 (backforecasts excluded)
              MS =  0.00005335  DF = 166

Modified Box-Pierce Chi-Square statistic

Lag             12      24      36      48
Chi-Square    11.5    19.5    27.8    41.1
DF              11      23      35      47
P-Value      0.399   0.670   0.801   0.716
```

(ii) State which ARIMA model has been fitted to the data and write out explicitly the equation of the fitted model for $\{Y_t\}$ in terms of an underlying white noise process $\{\varepsilon_t\}$.

(3)

(iii) State what approximate properties the residuals should have if the model is an adequate fit to the data and, in particular, how the acf of the residuals should behave. Without writing out any detailed formulae, explain how the Box-Pierce statistics are constructed and used to test the adequacy of the model. What conclusions can you draw in the present case?

(6)

(iv) Based upon the fitted model, the forecast value for the average exchange rate for January 2008 is 0.4846. Using the above output, find an approximate 95% prediction interval for the average exchange rate for January 2008, explaining your reasoning. In fact it turned out that the average exchange rate for January 2008 was 0.4785. Explain whether or not the size of the discrepancy between the observed and forecast values is surprising.

(5)

7. The Holt-Winters forecasting procedure is to be used for a time series with multiplicative seasonal variation of period p.

(i) Let Y_t denote the observed value of the series, L_t the local level, B_t the trend and I_t the seasonal index, all at time t. If α, γ and δ denote the smoothing constants for L_t, B_t and I_t respectively, write down the updating equations for L_t, B_t and I_t. (3)

(ii) Write down an expression for the forecast $\hat{y}_T(h)$ at time T for lead time h. (2)

The monthly sales of a product in thousands of litres are recorded over a number of years and Holt-Winters forecasting with multiplicative seasonal variation of period 12 is used. In the table below, the sales and other quantities that have been calculated month by month, using the smoothing constants $\alpha = 0.2$, $\gamma = 0.1$ and $\delta = 0.2$, are shown for the two years 1993 and 1994.

Year	Month	Sales	Level	Trend	Index	Fitted	Residual
1993	Jan	2075	3385.90	-7.21	0.678	2417.58	-342.58
1993	Feb	3264	3484.79	3.40	0.855	2821.09	442.91
1993	Mar	3308	3504.19	5.00	0.930	3233.84	74.16
1993	Apr	3688	3701.84	24.27	0.859	2893.73	794.27
1993	May	3136	3697.05	21.36	0.870	3263.22	-127.22
1993	Jun	2824	3637.17	13.24	0.837	3170.35	-346.35
1993	Jul	3644	3633.86	11.58	1.018	3728.51	-84.51
1993	Aug	4694	3846.17	31.66	1.052	3680.68	1013.32
1993	Sep	2914	3690.29	12.90	0.951	3843.34	-929.34
1993	Oct	3686	3685.23	11.11	1.016	3777.58	-91.58
1993	Nov	4358	3667.56	8.23	1.219	4534.51	-176.51
1993	Dec	5587	3755.22	16.17	1.395	5042.22	544.78
1994	Jan	2265	3685.66	7.60	0.665	2555.46	-290.46
1994	Feb	3685	3816.29	19.90	0.877	3158.84	526.16
1994	Mar	3754	3875.86	23.87	0.938	3569.46	184.54
1994	Apr	3708	3983.17	32.21	0.873	3349.65	358.35
1994	May	3210	3950.01	25.68	0.859	3494.46	-284.46
1994	Jun	3517	4020.56	30.16	0.845	3329.13	187.87
1994	Jul	3905	4008.01	25.89	1.009	4122.31	-217.31
1994	Aug	3670	3924.96	15.00	1.028	4242.95	-572.95
1994	Sep	4221	4039.84	24.99	0.970	3746.17	474.83
1994	Oct	4404	4118.69	30.37	1.027	4130.32	273.68
1994	Nov	5086	4153.67	30.83	1.220	5057.94	28.06
1994	Dec	5725	4168.42	29.22	1.391	5837.16	-112.16

(iii) Given the data as at December 1994, calculate forecasts of sales for (a) January 1995 and (b) December 1995. (4)

(iv) The sales in thousands of litres for January 1995 turned out to be 2367. Given this fact, calculate the values of all the remaining figures in the corresponding row of the table. (8)

(v) It was decided to investigate what might be the best set of values to choose for the smoothing constants α, γ and δ. Discuss how this might be done using the whole historical run of the above series. (3)

8. Let $\{Y_t\}$ be any ARMA process with autocorrelation function ρ_τ. The corresponding spectral density function $f(\omega)$ may be written as

$$f(\omega) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} \pi_\tau e^{-i\omega\tau} \qquad (-\pi \le \omega \le \pi).$$

(i) Stating any general properties of the autocorrelation function that you assume, show that $f(\omega)$ may be written equivalently as

$$f(\omega) = \frac{1}{2\pi}\left(1 + 2\sum_{\tau=1}^{\infty} \pi_\tau \cos\omega\tau\right) \qquad (-\pi \le \omega \le \pi).$$

Deduce that $f(\omega) = f(-\omega)$ $(-\pi \le \omega \le \pi)$ so that in considering $f(\omega)$ we may restrict attention to the range of values $0 \le \omega \le \pi$.

(4)

(ii) How you would interpret the spectral density function of any given ARMA process?

(2)

(iii) Write down $f(\omega)$ when $\{Y_t\}$ is a white noise process $\{\varepsilon_t\}$, and comment.

(3)

(iv) If $\{Y_t\}$ is an MA(1) process with model equation $Y_t = \varepsilon_t - \theta\varepsilon_{t-1}$, find the autocorrelation function and hence the spectral density function.

(5)

(v) Comment on how the shape of the spectral density function depends on the sign of θ and what this implies about the behaviour of the process, especially in comparison with a white noise process.

(6)