EXAMINATIONS OF THE ROYAL STATISTICAL SOCIETY
(formerly the Examinations of the Institute of Statisticians)

GRADUATE DIPLOMA, 2004

Statistical Theory and Methods II

Time Allowed: Three Hours

Candidates should answer **FIVE** *questions.*

All questions carry equal marks.
The number of marks allotted for each part-question is shown in brackets.

Graph paper and Official tables are provided.

Candidates may use silent, cordless, non-programmable electronic calculators.

Where a calculator is used the ***method*** *of calculation should be stated in full.*

The notation log *denotes logarithm to base* ***e****.*
Logarithms to any other base are explicitly identified, e.g. $\log_{10}$.

Note also that $\binom{n}{r}$ *is the same as* ${}^{n}C_{r}$.

This examination paper consists of 5 printed pages, **each printed on one side only**.
This front cover is page 1.
Question 1 starts on page 2.

There are 8 questions altogether in the paper.

1. Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with probability density function (pdf)

$$f(x) = \frac{x^{\theta-1}}{k_\theta \lambda^\theta} \exp(-x/\lambda), \qquad x > 0.$$

Here $\theta > 0$ and $\lambda > 0$ are parameters and k_θ is a normalising constant that depends only on θ. You are given that a random variable with this pdf has mean $\theta\lambda$ and variance $\theta\lambda^2$.

(i) Suppose that θ is known. Obtain $\tilde{\lambda}_\theta$, the method of moments estimator for λ, and show that it is unbiased for λ.

(4)

(ii) Show that the variance of $\tilde{\lambda}_\theta$ attains the Cramér-Rao lower bound.

(6)

(iii) Suppose instead that λ is known and θ is unknown. Show that $\sum_{i=1}^{n} \log X_i$ is sufficient for θ.

(5)

(iv) Find the method of moments estimator for θ when λ is known, and deduce from (iii) that the estimator is not fully efficient.

(5)

2. Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with probability density function (pdf)

$$f(x) = \frac{cx^{c-1}}{\lambda^c} \exp\left\{-\left(\frac{x}{\lambda}\right)^c\right\}, \qquad x > 0,$$

where c is a known positive constant and $\lambda > 0$ is a parameter.

(i) Let X be a random variable with the above pdf. By considering $\frac{\partial \log f}{\partial \lambda}$, or otherwise, show that $E(X^c) = \lambda^c$.

(5)

(ii) Obtain $\hat{\lambda}$, the maximum likelihood estimator of λ.

(5)

(iii) Find the Fisher information for λ and hence obtain the large sample variance of $\hat{\lambda}$.

(6)

(iv) Hence calculate an approximate 95% confidence interval for λ when $\hat{\lambda} = 4$, $c = 2$ and $n = 100$.

(4)

3. A random sample $X_1, X_2, \ldots, X_n$ is available from a Poisson distribution with mean θ. It is of interest to estimate λ, defined by $\lambda = e^{-\theta}$.

(i) Find the maximum likelihood estimator (MLE), $\hat{\theta}$, of θ. Hence deduce the MLE, $\hat{\lambda}$, of λ.

(6)

(ii) Find the variance of $\hat{\theta}$, and deduce the approximate variance of $\hat{\lambda}$ using the delta method.

(5)

(iii) An alternative estimator of λ is $\tilde{\lambda}$, defined as the observed proportion of zero observations. Find the bias of $\tilde{\lambda}$ and show that

$$\text{Var}\left(\tilde{\lambda}\right) = \frac{e^{-\theta}\left(1 - e^{-\theta}\right)}{n}.$$

(5)

(iv) Draw a rough sketch of the efficiency of $\tilde{\lambda}$ relative to $\hat{\lambda}$, and discuss its properties.

(4)

4. Observations $X_1, X_2, \ldots$ are available from a $N(\mu, 1)$ population. The null hypothesis H_0: $\mu = 0$ is to be tested against the alternative hypothesis H_1: $\mu = 1$.

(i) For a fixed sample size n, let $\overline{X}$ denote the mean of $X_1, X_2, \ldots, X_n$. Show that the Neyman-Pearson approach leads to rejection of H_0 in favour of H_1 when $\overline{X} > k$ for some suitable k.

(5)

(ii) Find the smallest value of n so that the Type I and Type II error probabilities are equal and are no more than 0.05.

(5)

(iii) Construct the sequential probability ratio test (SPRT) with approximately the same error probabilities. Calculate approximate expected sample sizes of the SPRT under H_0 and H_1.

(10)

5. Let $X_1, X_2, \ldots, X_n$ be a random sample from a continuous distribution with distribution function F. The null hypothesis is that $F(x) = F_0(x)$ for all x, where F_0 is a specified distribution function.

(i) Describe the Kolmogorov-Smirnov test.

(5)

(ii) Discuss briefly the strengths and weaknesses of this test.

(5)

(iii) The lifetimes in hours of a random sample of ten widgets from a large batch are as follows:

80, 57, 280, 15, 30, 251, 3, 45, 170, 145.

Perform a Kolmogorov-Smirnov test of the null hypothesis that the lifetimes have an exponential distribution with mean 100 hours. Report your conclusion.

(10)

6. A random sample $X_1, X_2, \ldots, X_n$ is available from a distribution indexed by a real parameter θ. Let T be an estimator of θ.

(i) Define the *risk* of T in terms of a specified loss function. What does it mean to say that T is an *inadmissible* estimator of θ?

(5)

(ii) Suppose now that this random sample comes from an exponential distribution with mean θ. Consider the maximum likelihood estimator (MLE) of θ, $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} X_i$, and the Pitman estimator of θ, $\tilde{\theta} = \frac{1}{n+1}\sum_{i=1}^{n} X_i$. Calculate the mean square errors of $\hat{\theta}$ and $\tilde{\theta}$.

(6)

(iii) Hence deduce that $\hat{\theta}$ is inadmissible with a squared error loss function.

(5)

(iv) Show that $\tilde{\theta}$ is a consistent estimator of θ.

(4)

Turn over

7. Observations $x_1, x_2, \ldots, x_n$ are available from a geometric distribution with probability mass function

$$p(x\,|\,\theta) = \theta(1-\theta)^x, \qquad x = 0, 1, 2, \ldots ,$$

where θ has a prior beta distribution with probability density function

$$p(\theta) \propto \theta^{a-1}(1-\theta)^{b-1}$$

for $0 < \theta < 1$ with parameters $a > 0$ and $b > 0$. [You are given that such a beta distribution with parameters a and b, as above, has mean $\dfrac{a}{a+b}$ and variance $\dfrac{ab}{(a+b+1)(a+b)^2}$.]

(i) Show that the beta distribution is a conjugate prior.

(6)

(ii) The prior belief about θ is that it has mean ½ and standard deviation $1/10$. Show that for the conjugate prior this implies that $a = b = 12$.

(4)

(iii) Suppose that $n = 88$ and $\sum x_i = 48$. Calculate the posterior mean and posterior standard deviation of θ.

(5)

(iv) Using an appropriate Normal approximation, calculate a Bayesian 90% posterior interval for θ.

(5)

8. A large random sample is available from a distribution indexed by a single real parameter, θ.

(i) Discuss the different ways in which the log-likelihood (viewed as a function of θ) may be used (a) to test H_0: $\theta = \theta_0$, for some specified θ_0, against H_1: $\theta \neq \theta_0$ and (b) to obtain a confidence interval for θ. You may assume that any necessary regularity conditions are satisfied.

(15)

(ii) Explain how the confidence interval in (b) above may be regarded as an approximation to a Bayesian interval for θ in certain circumstances.

(5)