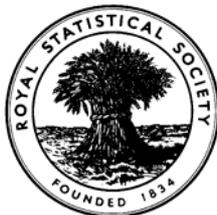


# EXAMINATIONS OF THE ROYAL STATISTICAL SOCIETY



## HIGHER CERTIFICATE IN STATISTICS, 2008

(Modular format)

### MODULE 5 : Further probability and inference

**Time allowed: One and a half hours**

*Candidates should answer **THREE** questions.*

*Each question carries 20 marks.*

*The number of marks allotted for each part-question is shown in brackets.*

*Graph paper and Official tables are provided.*

*Candidates may use calculators in accordance with the regulations published in the Society's "Guide to Examinations" (document Ex1).*

*The notation  $\log$  denotes logarithm to base  $e$ .*

*Logarithms to any other base are explicitly identified, e.g.  $\log_{10}$ .*

*Note also that  $\binom{n}{r}$  is the same as  ${}^n C_r$ .*

This examination paper consists of 3 printed pages **each printed on one side only**.

This front cover is page 1.

Question 1 starts on page 2.

There are 4 questions altogether in the paper.

1. Jane chooses a number  $X$  at random from the set of numbers  $\{1, 2, 3, 4\}$ , so that

$$P(X = k) = \frac{1}{4} \quad \text{for } k = 1, 2, 3, 4.$$

She then chooses a number  $Y$  at random from the subset of numbers  $\{X, \dots, 4\}$ ; for example, if  $X = 3$ , then  $Y$  is chosen at random from  $\{3, 4\}$ .

- (i) Find the joint probability distribution of  $X$  and  $Y$  and display it in the form of a two-way table. (5)
- (ii) Find the marginal probability distribution of  $Y$ , and hence find  $E(Y)$  and  $\text{Var}(Y)$ . (5)
- (iii) Show that  $\text{Cov}(X, Y) = 5/8$ . (5)
- (iv) Find the probability distribution of  $U = X + Y$ . (5)

2. Define the *probability generating function* and the *moment generating function* of a random variable  $X$  and give the relationship between these two functions. (3)

The random variable  $X$  has the binomial distribution with parameters  $n$  ( $n > 3$ ) and  $p$  ( $0 < p < 1$ ).

- (i) Show that the probability generating function of  $X$  is

$$\pi(t) = (pt + 1 - p)^n$$

for  $-\infty < t < \infty$ .

- (ii) Use part (i) to show that  $E(X) = np$  and  $\text{Var}(X) = np(1 - p)$ . (5)
- (iii) Find  $E(X^3)$ . (3)
- (iv) Now suppose that  $X_1, X_2, \dots, X_m$  are independent random variables and  $X_i$  has the binomial distribution with parameters  $n_i$  and  $p$  for  $i = 1, 2, \dots, m$ . Let  $Y = \sum_{i=1}^m X_i$ . Find the probability generating function of  $Y$ , and hence deduce the distribution of  $Y$ . (5)

3. A random sample of  $n$  independent observations  $X_1, X_2, \dots, X_n$  is taken from a population which has probability density function

$$f(x) = \frac{xe^{-x/\lambda}}{\lambda^2}, \quad x > 0,$$

where  $\lambda$  ( $\lambda > 0$ ) is an unknown parameter. The sample mean is denoted by  $\bar{X}$ .

- (i) Show that  $\hat{\lambda} = \bar{X}/2$  is the method of moments estimator of  $\lambda$ . (6)

- (ii) Show that  $\hat{\lambda}$  is an unbiased estimator of  $\lambda$  and find  $\text{Var}(\hat{\lambda})$ . Hence deduce that  $\hat{\lambda}$  is a consistent estimator of  $\lambda$ . (9)

- (iii) Suppose that  $n = 3$  and the alternative estimator

$$\tilde{\lambda} = \frac{1}{8}X_1 + \frac{1}{4}X_2 + \frac{1}{8}X_3$$

has been proposed. Find the relative efficiency of this estimator compared to  $\hat{\lambda}$  and say, with reasons, which estimator you prefer.

(5)

4. A random sample  $X_1, X_2, \dots, X_n$  is drawn from the Normal distribution with mean 0 and variance  $\theta$ .

- (i) Obtain the likelihood function. (5)

- (ii) Find the maximum likelihood estimator,  $\hat{\theta}$ , of  $\theta$ . (6)

- (iii) Using a large sample property of maximum likelihood estimators, find the approximate distribution of  $\hat{\theta}$  when  $n$  is large. (5)

- (iv) Find an approximate 95% confidence interval for  $\theta$  when  $n = 100$  and  $\sum X_i^2 = 1000$ . (4)