EXAMINATIONS OF THE ROYAL STATISTICAL SOCIETY

HIGHER CERTIFICATE IN STATISTICS, 2009

MODULE 2 : Probability models

Time allowed: One and a half hours

Candidates should answer **THREE** *questions.*

Each question carries 20 marks.
The number of marks allotted for each part-question is shown in brackets.

Graph paper and Official tables are provided.

Candidates may use calculators in accordance with the regulations published in the Society's "Guide to Examinations" (document Ex1).

The notation log *denotes logarithm to base* ***e***.
Logarithms to any other base are explicitly identified, e.g. $\log_{10}$.

Note also that $\binom{n}{r}$ *is the same as* ${}^{n}C_{r}$.

This examination paper consists of 4 printed pages **each printed on one side only**.
This front cover is page 1.
Question 1 starts on page 2.

There are 4 questions altogether in the paper.

1. A standard pack of 52 playing cards consists of 4 suits (clubs, diamonds, hearts and spades), each consisting of 13 cards numbered 2, 3, 4, …, 10, Jack, Queen, King, Ace (their face values). In the game of poker, a hand of 5 cards is drawn without replacement from a well-shuffled pack.

(i) How many different poker hands are possible?

(4)

(ii) A poker hand consisting of a pair of cards with the same face value and three other cards with the same face value (different from that of the pair) is called a *full house*. Find the probability that a poker hand drawn from a well-shuffled pack is a full house. Express your answer either as a fraction in lowest terms or as a decimal correct to 3 significant figures.

(8)

(iii) A poker hand consisting of a pair of cards with the same face value and three other cards with face values different from each other and from that of the pair is called a *pair*. Find the probability that a poker hand drawn from a well-shuffled pack is a pair. Express your answer either as a fraction in lowest terms or as a decimal correct to 3 significant figures.

(8)

2. A blended wine is intended to comprise two parts of Sauvignon to one part of Merlot. The amounts dispensed to make up a nominal 75cl bottle of this wine are X cl of Sauvignon and Y cl of Merlot, where X and Y are assumed to be independent Normally distributed random variables with respective means 52 and 26 cl and respective variances 1 and 0.5625.

(i) Find the probability that the actual volume of wine dispensed into a bottle is less than the nominal volume.

(6)

(ii) Find the distribution of $X - 2.2Y$, and use this distribution to find the probability that the ratio of Sauvignon to Merlot is greater than 2.2. By a similar method, find the probability that this ratio is less than 1.8. Hence state to three decimal places the probability that the ratio of Sauvignon to Merlot in a randomly chosen bottle differs from 2 to 1 by more than 10%.

(10)

(iii) Based on your final answer to part (ii), and assuming that bottles are filled independently, write down (a) the exact distribution, (b) a suitable approximation, for the number of bottles in a thousand in which the ratio of Sauvignon to Merlot differs from 2 to 1 by more than 10%. Hence find the approximate probability that there are 10 or more bottles in a consignment of 1000 in which the ratio of Sauvignon to Merlot differs from 2 to 1 by more than 10%, giving your answer to three decimal places.

(4)

3. The discrete random variable X has the Poisson distribution with parameter λ, so that its probability mass function is given by

$$p_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots, \quad \lambda > 0.$$

(i) Given that $E(X) = \lambda$, show that $E(X^2) = \lambda(\lambda + 1)$ and deduce $\text{Var}(X)$.

(5)

(ii) The random variables Y and Z have Poisson distributions with respective parameters μ and λ, and X, Y and Z are independent. Use the relation

$$P(W = w) = \sum_{x=0}^{w} P(X = x)P(Y = w - x)$$

to show that the random variable $W = X + Y$ has a Poisson distribution, and write down its mean and variance. What is the distribution of the random variable $V = Y + Z$?

(6)

(iii) The random variables T and U are defined by $T = W - Z$, $U = V - Z$. Find $E(T)$, $\text{Var}(T)$, $E(U)$ and $\text{Var}(U)$. Explain why $P(U < 0) = 0$ but $P(T < 0) > 0$.

(9)

Turn over

4. The continuous random variable X has probability density function (pdf)

$$f_X(x) = \frac{1}{2\theta}, \quad -\theta \le x \le \theta.$$

(i) Show that the cumulative distribution function of X is

$$F_X(x) = \frac{\theta + x}{2\theta}, \quad -\theta \le x \le \theta.$$

Deduce an expression for $P(X > x)$.

(4)

(ii) The random variable Y has the same distribution as X and is independent of X. The random variable Z is defined as $\max(X, Y)$. Show that

$$P(Z \le z) = \left(\frac{\theta + z}{2\theta}\right)^2, \quad -\theta \le z \le \theta.$$

Deduce the pdf of Z and hence find $E(Z)$.

(7)

(iii) The random variable W is defined as $\min(X, Y)$. Show that

$$P(W > w) = \left(\frac{\theta - w}{2\theta}\right)^2, \quad -\theta \le w \le \theta.$$

Hence find the pdf of W and $E(W)$.

(7)

(iv) Find the value of the constant k such that $E[k(Z - W)] = \theta$.

(2)