EXAMINATIONS OF THE ROYAL STATISTICAL SOCIETY

GRADUATE DIPLOMA, 2010

MODULE 3 : Stochastic processes and time series

Time Allowed: Three Hours

Candidates should answer **FIVE** *questions.*

All questions carry equal marks.
The number of marks allotted for each part-question is shown in brackets.

Graph paper and Official tables are provided.

Candidates may use calculators in accordance with the regulations published in the Society's "Guide to Examinations" (document Ex1).

The notation log *denotes logarithm to base* ***e***.
Logarithms to any other base are explicitly identified, e.g. $\log_{10}$.

Note also that $\binom{n}{r}$ *is the same as* ${}^{n}C_{r}$.

This examination paper consists of 9 printed pages, **each printed on one side only**.
This front cover is page 1.
Question 1 starts on page 2.

There are 8 questions altogether in the paper.

1. In the control of industrial processes, a Shewhart control chart has warning limits and action limits, where the action limits are outside the warning limits. A sequence of independent observations of the process is plotted on the chart, where each observation is either (a) within the warning limits, which happens with probability $1 - \theta$, or (b) between the warning and action limits, which happens with probability $\theta - \phi$, or (c) outside the action limits, which happens with probability ϕ. Here θ and ϕ are parameters which satisfy $0 < \phi < \theta < 1$. The process is stopped as soon as either an observation falls outside the action limits or two successive observations fall outside the warning limits.

Consider modelling the process being monitored by the Shewhart control chart as a Markov chain with three states, 0, 1 and 2. State 0 corresponds to the current process value being within the warning limits. State 1 corresponds to the current process value being between the warning and action limits, the previous process value having been within the warning limits. State 2 corresponds to the process being stopped, so is an absorbing state.

(i) Write down the corresponding transition matrix in terms of the parameters θ and ϕ.

(5)

(ii) For $i = 0, 1$, let x_i denote the expected number of further observations until the process is stopped, given that the process is currently in state i. By conditioning on the outcome of the next observation, write down a pair of backward equations for x_0 and x_1.

(5)

(iii) Solve the backward equations to obtain explicit expressions for x_0 and x_1, showing in particular that

$$x_0 = \frac{1+\theta-\phi}{\theta^2+\phi(1-\theta)}.$$

(5)

(iv) Suppose that the warning and action limits are chosen such that the probability that an observation falls outside the warning limits is $\frac{1}{20}$ and the probability that an observation falls outside the action limits is $\frac{1}{100}$. For a process that has just started to be monitored, evaluate to the nearest integer the expected number of observations until the process is stopped.

(5)

2. In a television quiz show, a contestant is asked a succession of questions. Assume that for each question, independently of the answers to any other questions, the probability of a correct answer is θ and the probability of an incorrect answer is $1 - \theta$, where $0 < \theta < 1$. At the end of the game, the prize depends on the number of successive correct answers given since the last incorrect answer or, if all have been answered correctly, the number of questions. Let $\{X_n\}$ $(n \geq 0)$ denote a Markov chain in which X_n represents the situation after the nth question, namely the number of correct answers since the last incorrect answer, with $X_0 = 0$ assumed by convention. After each question the number of correct answers since the last incorrect answer either increases by one or falls to zero.

(i) Assuming first that there is no upper bound on the number of questions that will be asked, write down the state space S and the transition probabilities $\{p_{ij}\}$ for the Markov chain $\{X_n\}$.

(4)

(ii) Show that the stationary distribution $\{\pi_i\}$ for $\{X_n\}$ is given by

$$\pi_i = (1-\theta)\theta^i \quad (i \geq 0).$$

(6)

(iii) Suppose now that the contestant is to be asked exactly six questions. By calculating the six-step transition probabilities $p_{0j}^{(6)}$, find $P(X_6 = j)$ for $j = 0, 1, \ldots, 6$.

(6)

(iv) The contestant receives a prize of £50×2^j if $X_6 = j$ for $j \geq 1$. Calculate the expected winnings when $\theta = \frac{1}{2}$.

(4)

3. Suppose that two players, A and B, are playing a simple game against each other. A sequence of plays, whose results are independent of each other, with a stake of £1 per play, continues until one or other of the players loses all his capital, i.e. is ruined. Player A starts with capital £a and Player B starts with capital £b, where a and b are positive integers. Let $N = a + b$, the total capital in pounds of both players combined. At each play, Player A has probability θ of winning, where $0 < \theta < 1$, and Player B has probability $1 - \theta$ of winning. At each play, £1 is transferred from the loser to the winner.

(i) Let x_i denote the probability that Player A is eventually ruined, given that currently he has capital £i remaining, where $0 \le i \le N$.

(a) Write down a difference equation satisfied by the x_i, together with appropriate boundary conditions.

(2)

(b) Assuming that $\theta \neq \frac{1}{2}$, solve the difference equation of part (a) together with its boundary conditions.

(6)

(c) Deduce that the probability at the start of the game that Player A is eventually ruined is given by

$$\frac{\left(\frac{1-\theta}{\theta}\right)^N - \left(\frac{1-\theta}{\theta}\right)^a}{\left(\frac{1-\theta}{\theta}\right)^N - 1}.$$

(1)

(ii) Suppose as before that Player A starts with capital £a but that Player B has unlimited resources.

(a) By letting $N \to \infty$ in the result of part (i)(c), find the probability that Player A is eventually ruined, distinguishing between the two cases $0 < \theta < \frac{1}{2}$ and $\frac{1}{2} < \theta < 1$.

(5)

(b) Find the probability that Player A is ruined in the case $\theta = \frac{1}{2}$.

(6)

4. Consider the M/M/∞ model for an infinite server queue, in which customers arrive according to a Poisson process with rate λ. There is an unlimited number of servers available, and service times are independently and identically distributed, having an exponential distribution with mean $\dfrac{1}{\mu}$.

(i) For the corresponding continuous time Markov chain, $\{N(t)\}$ $(t \geq 0)$, specify the state space and write down the instantaneous transition rates.

(3)

Assume that the queue is empty at time 0. Define

$$p_n(t) = P\big(N(t) = n \mid N(0) = 0\big) \qquad (n \geq 0)$$

and

$$G(z,t) = \sum_{n=0}^{\infty} p_n(t) z^n .$$

(ii) Derive the forward equations for the $p_n(t)$ $(n \geq 0)$.

(6)

(iii) Deduce that $G(z, t)$ satisfies the partial differential equation

$$\frac{\partial G}{\partial t} = (z-1)\left(\lambda G - \mu \frac{\partial G}{\partial z}\right).$$

(5)

(iv) Verify that the solution of the partial differential equation with the appropriate initial condition is given by

$$G(z,t) = \exp\left[\theta(t)(z-1)\right],$$

where $\theta(t) = \dfrac{\lambda}{\mu}\left(1 - \mathrm{e}^{-\mu t}\right)$ $(t \geq 0)$, and identify the distribution of $N(t)$ for all $t \geq 0$.

(6)

Turn over

5. A checkout counter in a shop always has a cashier present. Customers arrive at the checkout according to a Poisson process with rate λ per hour. When there is only a single customer at the counter, the cashier works alone with a service rate of α per hour. Whenever there is more than one customer at the checkout, however, a "bagger" is added, increasing the service rate to β per hour.

(i) Assuming a continuous time Markov chain model for the number of customers at the checkout, write down the state space and transition rates.

(4)

(ii) Write down the detailed balance equations and find the equilibrium distribution for the chain, stating what condition has to be satisfied by the parameters for an equilibrium distribution to exist. In particular, show that the long-term proportion of time that the queue at the checkout is empty, i.e. the cashier is free, is given by

$$\frac{\alpha(\beta-\lambda)}{\alpha(\beta-\lambda)+\beta\lambda}.$$

(12)

(iii) A customer arrives to find k customers ahead of him in the queue, where $k \geq 1$.

State the distribution of the residual service time of the customer currently being served, and also the distribution of the service time of each of the remaining customers ahead of the newly arrived customer, including a specification of any parameter or parameters.

State also the distribution of the length of time that the newly arrived customer has to wait before he starts being served, including a specification of any parameter or parameters.

[There is no need to write out the probability density functions of the distributions.]

(4)

6. Consider the AR(2) model

$$Y_t = \tfrac{1}{2}Y_{t-1} - \tfrac{3}{64}Y_{t-2} + \varepsilon_t \qquad (-\infty < t < \infty)$$

for a process $\{Y_t\}$, where $\{\varepsilon_t\}$ is a white noise process.

(i) Find the roots of the autoregressive characteristic equation and check that the stationarity condition is satisfied.

(4)

(ii) Find the Yule-Walker equations that are satisfied by the autocorrelation function $\{\rho_\tau\}$.

(4)

(iii) Obtain the value of ρ_1.

(3)

(iv) Show that a general expression for the autocorrelation function is given by

$$\rho_\tau = \tfrac{189}{134}\left(\tfrac{3}{8}\right)^\tau - \tfrac{55}{134}\left(\tfrac{1}{8}\right)^\tau \qquad (\tau \geq 0)\,.$$

(9)

Turn over

7. (i) Define the terms *strict stationarity* and *weak stationarity* in the context of time series analysis.

(2)

(ii) Specify fully the model for an MA(1) process, and derive the mean, variance and autocorrelation function (ACF) of an MA(1) process in terms of the parameters of your model.

(7)

(iii) Explain what a partial autocorrelation coefficient measures. State, without giving detailed working, the form of the partial autocorrelation function (PACF) for an MA(1) process.

(3)

(iv) Describe how you would use the sample ACF and PACF to recognise an MA(1) process.

(2)

(v) The following three tables are derived from 50 observations from each of three time series, Series A, B and C. Which one of the three do you think is an MA(1) process? Justify your answer.

Suggest models for the other two time series, giving reasons for your answers.

(6)

Series A

lag	*acf*	*pacf*
1	–0.84	–0.84
2	0.74	0.13
3	–0.63	0.10
4	0.53	–0.00
5	–0.44	0.04
6	0.41	0.16
7	–0.36	0.04
8	0.26	–0.23
9	–0.18	0.06
10	0.09	–0.03
11	–0.03	–0.02
12	–0.04	–0.07

Series B

lag	*acf*	*pacf*
1	0.42	0.42
2	–0.13	–0.37
3	–0.09	0.19
4	–0.02	–0.15
5	0.05	0.18
6	0.00	–0.18
7	–0.07	0.08
8	–0.09	–0.17
9	–0.15	–0.05
10	–0.10	–0.04
11	0.02	0.05
12	0.04	–0.04

Series C

lag	*acf*	*pacf*
1	0.04	0.04
2	–0.08	–0.08
3	0.05	0.06
4	–0.08	–0.10
5	–0.10	–0.09
6	0.04	0.03
7	–0.02	–0.03
8	–0.04	–0.03
9	–0.02	–0.04
10	–0.23	–0.24
11	0.08	0.11
12	–0.03	–0.10

Turn over

8. Let Y_t denote the observed value of a time series at time t and $\hat{y}_T(h)$ the forecast at time T for lead time h.

(i) Holt's two-parameter smoothing method is to be used for forecasting. Let L_t denote the local level and B_t the trend at time t. If α and γ denote the smoothing constants for L_t and B_t, respectively, write down (a) the updating equations for L_t and B_t, and (b) an expression for $\hat{y}_T(h)$.

(3)

In an industrial experiment, readings of the temperature (in °C) of a chemical process are taken at regular intervals of one minute. Holt's method is adopted for forecasting, using the smoothing constants α = 0.7 and γ = 0.8. The following table gives the calculations for a segment of ten successive observations at times labelled t = 1, 2, …, 10.

Time *t*	*Temp.* *(°C)*	*Level*	*Trend*	*Fitted* *(forecast)*	*Residual* *(error)*	*Squared* *residual*
1	21.7	21.682	0.285	21.640	0.060	0.004
2	21.8	21.850	0.192	21.967	–0.167	0.028
3	21.9	21.943	0.112	22.042	–0.142	0.020
4	22.2	22.156	0.194	22.055	0.145	0.021
5	22.5	22.455	0.278	22.350	0.150	0.023
6	22.8	22.780	0.315	22.733	0.067	0.005
7	23.1	23.099	0.318	23.095	0.005	0.000
8	23.4	23.405	0.309	23.417	–0.017	0.000
9	23.8	23.774	0.357	23.714	0.086	0.007
10	24.1	24.109	0.340	24.131	–0.031	0.001

(ii) Explain what the "fitted (forecast)" and "residual (error)" values are in the output, illustrating your explanation by showing how the values for time t = 10 have been calculated.

(4)

(iii) Given the data up to time t = 10, obtain the forecast temperature (to 3 decimal places) for the next three time points.

(3)

(iv) The temperature for t = 11 turned out to be 24.6. Given this fact, calculate (to 3 decimal places) the values of "level" and "trend" in the corresponding row of the output.

(4)

(v) Given a historical run of a time series, to which Holt's method has been applied, define the *mean absolute deviation* (MAD) and the *mean square deviation* (MSD). Illustrate the definitions by calculating the MAD and the MSD using the data given above for the 10 time points.

(4)

(vi) Given a historical run of a time series, explain how an optimal choice of the smoothing constants α and γ might be made using the MAD or the MSD.

(2)