This examination paper consists of 3 printed pages, each printed on one side only.
This front cover is page 1.
Question 1 starts on page 2.

There are 4 questions altogether in the paper.

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1. Two tennis players, A and B, are playing a match. Let $X$ be the number of serves faster than 125 mph served by A in one of his service games and let $Y$ be the number of these serves returned by B. The following probability model is proposed:

$$P(X = 0) = 0.4, \quad P(X = 1) = 0.3, \quad P(X = 2) = 0.2 \quad \text{and} \quad P(X = 3) = 0.1.$$ 

The conditional distribution of $Y$ (given that $X > 0$) is binomial with parameters $x$ and 0.4, and $P(Y = 0 \mid X = 0) = 1$. Assume that this model is correct when answering the following questions.

(i) Find the joint probability distribution of $X$ and $Y$ and display it in the form of a two-way table. 

(ii) Find the marginal distribution of $Y$ and evaluate $E(Y)$. 

(iii) Find Cov($X, Y$). 

(iv) Use your joint probability distribution table to find the probability distribution of the number of serves faster than 125 mph that are not returned by B in a game. 

2. The joint probability density function of the random variables $X$ and $Y$ is

$$f(x, y) = \frac{1}{2\pi} \exp\left(-\frac{1}{4}(x - 1)^2 - \frac{1}{4}(y - (1 + x))^2\right), \quad -\infty < x < \infty, \quad -\infty < y < \infty.$$ 

(i) Use integration to show that $X$ has the Normal distribution with mean 1 and variance 2. 

(ii) Use integration to show that the moment generating function of $X$ is $m_X(t) = \exp(t + t^2)$. 

(iii) Use the moment generating function to find $E(X^3)$. 

2

Turn over
3. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with probability density function

$$f(x) = \beta(1-x)^{\beta-1}, \quad 0 < x < 1,$$

where $\beta (> 0)$ is an unknown parameter.

(i) Find the maximum likelihood estimator, $\hat{\beta}$, of $\beta$.

(ii) Calculate the approximate variance of $\hat{\beta}$ and use it to determine an approximate 95% confidence interval for $\beta$ when $n$ is large.

(iii) Show that $P(X_i < 0.5) = 1 - 0.5^\beta$.

(iv) Suppose now that the values of $X_1, X_2, \ldots, X_n$ are not known, but you do know $Y$, the number of the $X_i$ less than 0.5. State the distribution of $Y$, and write down the likelihood function of $\beta$ based on $Y$.

4. (a) Define the bias, relative efficiency and efficiency of potential estimators of a population parameter, and explain briefly why these are useful when deciding between different estimators.

(b) The random variables $Y_1, Y_2, \ldots, Y_n$ constitute a random sample from a discrete distribution, with $P(Y_i = -2) = \frac{1}{2}(1-p)^2$, $P(Y_i = 0) = p(1-p)$, $P(Y_i = 1) = p$, $P(Y_i = 2) = \frac{1}{2}(1-p)^2$ and $P(Y_i = k) = 0$ for $k \neq -2, 0, 1, 2$, where $p (0 < p < 1)$ is an unknown parameter.

(i) Find the method of moments estimator, $\tilde{p}$, of $p$. Describe one unsatisfactory feature that this estimator possesses.

(ii) Find $E(Y_i^2)$ and hence obtain the variance of $\tilde{p}$.

(iii) Making use of results found in answering parts (i) and (ii), or otherwise, find an unbiased estimator of $p^2$ based on $\sum Y_i$ and $\sum Y_i^2$. 
