EXAMINATIONS OF THE ROYAL STATISTICAL SOCIETY

GRADUATE DIPLOMA, 2012

MODULE 1 : Probability distributions

Time allowed: Three Hours

Candidates should answer **FIVE** *questions.*

All questions carry equal marks.
The number of marks allotted for each part-question is shown in brackets.

Graph paper and Official tables are provided.

Candidates may use calculators in accordance with the regulations published in the Society's "Guide to Examinations" (document Ex1).

The notation log *denotes logarithm to base* ***e***.
Logarithms to any other base are explicitly identified, e.g. $\log_{10}$.

Note also that $\binom{n}{r}$ *is the same as* ${}^{n}C_{r}$.

This examination paper consists of 8 printed pages.
This front cover is page 1.
Question 1 starts on page 2.

There are 8 questions altogether in the paper.

1. An experiment has n possible outcomes labelled 1, 2, …, n, each with probability $\frac{1}{n}$, for some $n > 1$.

(a) Suppose that $n = 8$, and $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 5, 6\}$ and $C = \{1, 3, 7, 8\}$.

(i) Show that $P(A \cap B) = P(A)P(B)$, and that $P(A \cap B \cap C) = P(A)P(B)P(C)$.
(4)

(ii) By considering another intersection, show that A, B and C are not independent.
(2)

(iii) Construct an event D so that A, B and D are independent.
(4)

(b) By considering the cases $n = 6m + r$ for $r = 0, 1, 2, 3, 4, 5$, find the values of n for which the events

E = {Outcome divisible by 2} and F = {Outcome divisible by 3}

are independent.
(10)

2. Suppose X has a Poisson distribution with parameter λ.

(i) Given an integer $r \geq 1$, show that $E(X(X-1)(X-2)\ldots(X-r+1)) = \lambda^r$. Deduce the coefficient of skewness of X.
(8)

[The coefficient of skewness of a random variable Y with mean μ and variance $\sigma^2 > 0$ is $\dfrac{E(Y-\mu)^3}{\sigma^3}$.]

(ii) Find $E\left(\dfrac{1}{X+1}\right)$.
(4)

(iii) Suppose first that $\lambda > 1$ is not an integer. Show that the sequence $p_0, p_1, p_2, \ldots$, where p_k denotes the probability $P(X = k)$, increases to a unique maximum, then decreases. Describe the behaviour of this sequence when $\lambda \geq 1$ is an integer.
(8)

3. (i) A random variable has the continuous uniform distribution over the interval $[a, b]$. Write down its mean and show that its variance is $\frac{(b-a)^2}{12}$.

(6)

(ii) Let X be a random variable with mean μ, variance σ^2, and $0 \le X \le 1$. Given X, the independent random variables V and W are uniformly distributed over the intervals $[0, X]$ and $[X, 1]$ respectively. Write $Y = W - V$. Find the mean and variance of Y, in terms of μ and σ^2.

(14)

[You may use the results

$$E(Y) = E(E(Y \mid X)) \quad \text{and} \quad \text{Var}(Y) = E(\text{Var}(Y \mid X)) + \text{Var}(E(Y \mid X)).]$$

4. A sequence of Bernoulli trials is conducted, in which the probability of success is $p > 0$. Let W denote the number of trials needed to obtain the first success. Write down the distribution of W, and find its probability generating function (pgf). Hence or otherwise show that its mean and variance are $\frac{1}{p}$ and $\frac{1-p}{p^2}$ respectively.

(10)

Deduce the mean and variance of the number of trials needed to obtain k successes, where $k \ge 2$.

(6)

By using pgfs or otherwise, find the probability that exactly n trials are required to obtain k successes.

(4)

[You may quote standard properties of pgfs without proof.]

5. (i) Let U have a continuous uniform distribution over the interval [0, 1], and let $f(.)$ be a continuous function defined on that interval. Write $\mu = E(f(U))$ and $\sigma^2 = \text{Var}(f(U))$. Show that $\mu = \int_0^1 f(u)\,du$ and $\sigma^2 = \int_0^1 (f(u))^2\,du - \mu^2$.

(2)

(ii) Let $g(U) = \dfrac{f(U) + f(1-U)}{2}$. Deduce that

$$E(g(U)) = \mu, \quad \text{and that} \quad \text{Var}(g(U)) = \frac{\sigma^2 + \tau}{2},$$

where τ is the covariance of $f(U)$ and $f(1-U)$.

(4)

(iii) Consider the particular case when $f(U) = \dfrac{1}{1+U}$. You are given that $\mu = \log 2$.

(a) Evaluate τ.

(6)

(b) Let $\{U_i : i = 1, 2, \ldots, 2n\}$ be independent, all having the same distribution as U. Write $A_n = \dfrac{\sum_{i=1}^{2n} f(U_i)}{2n}$ and $B_n = \dfrac{\sum_{i=1}^{n} g(U_i)}{n}$.

Show that $E(A_n) = E(B_n) = \log 2$, and evaluate the ratio $\dfrac{\text{Var}(B_n)}{\text{Var}(A_n)}$ to two significant figures, given that $\sigma^2 = \frac{1}{2} - (\log 2)^2$.

(8)

6. Let the random variable Z have a standard Normal distribution. You are given that its moment generating function (mgf) is $\exp\left(\frac{1}{2}t^2\right)$, and that of Z^2 is $(1-2t)^{-\frac{1}{2}}$ for $t<\frac{1}{2}$.

(i) Use the mgf of Z^2 to find the mean and variance of Z^2.

(4)

(ii) Let Z_1, Z_2, Z_3, … be independent random variables, all having the standard Normal distribution, and define $W_n = Z_1^2 + Z_2^2 + \ldots + Z_n^2$. Show that $E(W_n) = n$, $\text{Var}(W_n) = 2n$, and find $g_n(t)$, the mgf of $X_n = \dfrac{W_n - n}{\sqrt{2n}}$.

(8)

(iii) Show that $\log(g_n(t))$ converges to $\frac{1}{2}t^2$ as $n \to \infty$. Hence deduce the approximate distribution of W_n for large n, and thus, approximately, the probability that W_{50} exceeds 60.

(8)

7. (i) Let U have the continuous uniform distribution over the interval [0, 1]. Show that $X = -n\log U$ has the exponential distribution with probability density function $g(x) = \dfrac{\exp(-x/n)}{n}$ on $x \geq 0$.

(4)

(ii) The density function of a random variable having a gamma distribution is given by $f(x) = \dfrac{e^{-x}x^{n-1}}{(n-1)!}$ on $x \geq 0$ for integer $n > 1$. Show that $\dfrac{f(x)}{g(x)}$ reaches its maximum when $x = n$, and that the maximum value is $k = \dfrac{n^n e^{1-n}}{(n-1)!}$.

(10)

[Hint: Take the log of $\dfrac{f(x)}{g(x)}$.]

(iii) You have a supply of values from independent random variables U_1, U_2, U_3, …, all having the same distribution as U in part (i). Describe how to use them, as well as the results of parts (i) and (ii), in a rejection method to generate a stream of values from a random variable having the density $f(x)$ given in part (ii).

(6)

8. Let C be a circle with unit radius so that the length of the side of an inscribed equilateral triangle is $\sqrt{3}$. Three possible ways of describing a "random chord" are suggested.

(i) Select two points independently and uniformly distributed on the circumference, and join them.

(ii) Select a point P within the interior of the circle at random (i.e. uniformly distributed), and join P to the centre of the circle along a radius. The chord is the line through P perpendicular to this radius.

(iii) First select a radius at random (i.e. uniformly distributed over all directions), then choose a point Q uniformly distributed along this radius. The chord is the line through Q perpendicular to this radius.

In each case, find the probability that the length of the "random chord" exceeds $\sqrt{3}$.

(8, 6, 6)

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