EXAMINATIONS OF THE ROYAL STATISTICAL SOCIETY

GRADUATE DIPLOMA, 2013

MODULE 1 : Probability distributions

Time allowed: Three Hours

Candidates should answer **FIVE** *questions.*

All questions carry equal marks.
The number of marks allotted for each part-question is shown in brackets.

Graph paper and Official tables are provided.

Candidates may use calculators in accordance with the regulations published in the Society's "Guide to Examinations" (document Ex1).

The notation log *denotes logarithm to base* ***e***.
Logarithms to any other base are explicitly identified, e.g. $\log_{10}$.

Note also that $\binom{n}{r}$ *is the same as* ${}^nC_r$.

This examination paper consists of 12 printed pages.
This front cover is page 1.
Question 1 starts on page 2.

There are 8 questions altogether in the paper.

1. (a) The events $E_1, E_2, \ldots, E_n$ form a mutually exclusive and exhaustive partition of the sample space S. Another event A in S has probability $P(A) > 0$. Write down the *Law of Total Probability*, which expresses $P(A)$ in terms of conditional and unconditional probabilities involving the events $E_1, E_2, \ldots, E_n$. Write down *Bayes' Theorem* for probabilities of the form $P(E_j \mid A)$.

(4)

(b) Each question in a certain multiple-choice examination has 4 possible answers, of which just 1 is correct. It can be assumed that candidates who do not know the correct answer to a question always guess it, choosing one of the 4 possible answers at random.

(i) A particular candidate has probability θ ($0 < \theta < 1$) of knowing the correct answer to a question. Show that the probability that this candidate answers a question correctly is $\frac{1}{4}(1+3\theta)$.

(4)

(ii) When a candidate gives the correct answer, 1 mark is awarded. When a candidate gives the wrong answer, a fraction $\dfrac{1}{n}$ of a mark is **deducted**. Show that the value $n = 3$ makes the expected number of marks awarded to the candidate in part (i) equal to θ.

(6)

(iii) The examination consists of 60 questions. Verify that, when $n = 3$ as in part (ii), this candidate must give at least 45 correct answers in order to obtain a total of at least 40 marks.

Assume also that, in a particular case, $\theta = 0.75$ independently for each question. Find approximately the probability that this candidate's total mark for the examination is at least 40.

(6)

2. For some $\theta > 0$, the continuous random variable X has the probability density function $f(x) = \theta e^{-\theta x}$ $(x > 0)$, i.e. X has the exponential distribution with expected value $\frac{1}{\theta}$ and variance $\frac{1}{\theta^2}$. Let $Y = \sqrt{X}$.

(i) Use Taylor series expansions to show that $\frac{7}{8\sqrt{\theta}}$ and $\frac{1}{4\theta}$ are approximate expressions for $E(Y)$ and Var(Y) respectively.

(6)

(ii) Prove that Y has the Weibull distribution, with probability density function $g(y) = 2\theta y \exp(-\theta y^2)$ $(y > 0)$.

(7)

(iii) Hence obtain exact expressions for $E(Y)$ and Var(Y), and compare these with the approximations obtained in part (i).

[Hint: the gamma function is defined by $\Gamma(k) = \int_0^\infty u^{k-1} e^{-u}\, du$ for $k > 0$; you may use, without proof, the result that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.]

(7)

3. The independent continuous random variables $X_1, X_2, \ldots, X_n$ (for $n \geq 2$) are identically distributed, each with cumulative distribution function $F(x)$ and probability density function $f(x)$.

(i) Let $V = \min(X_1, X_2, \ldots, X_n)$. Explain why, for any value v,

$$P(V \leq v) = 1 - \{1 - F(v)\}^n .$$

Hence write down the probability density function of V.

(3)

(ii) Let $W = \max(X_1, X_2, \ldots, X_n)$. Find the cumulative distribution function and probability density function of W in terms of $f(w)$ and $F(w)$.

(3)

(iii) Explain why, for any values v and w such that $v \leq w$,

$$P(V \leq v \text{ and } W \leq w) = P(W \leq w) - P(V > v \text{ and } W \leq w)$$

and why

$$P(V > v \text{ and } W \leq w) = [F(w) - F(v)]^n .$$

Hence show that the joint probability density function of V and W is

$$f_{VW}(v, w) = n(n-1) f(v) f(w)[F(w) - F(v)]^{n-2}, \quad v \leq w.$$

(6)

(iv) Suppose now that each X_i, $i = 1, 2, \ldots, n$, has the uniform distribution on the interval (0, 1). Show that $E(VW) = \dfrac{1}{n+2}$ and find the covariance of V and W.

[Hint: you may use, without proof, the result that, for non-negative integers r and s, $\int_0^1 u^r (1-u)^s \, du = \dfrac{r!s!}{(r+s+1)!}$.]

(8)

4. (a) The random variables X_1, X_2 have the bivariate Normal distribution with expectation $\boldsymbol{\mu} = (\mu_1 \quad \mu_2)^T$ and covariance matrix $\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$.

(i) Write out explicitly the joint probability density function of X_1 and X_2.

(3)

(ii) State (without proof) the marginal distribution of X_2 and write out its marginal probability density function.

(1)

(iii) Hence obtain the conditional probability density function of X_1 given that $X_2 = x_2$. Identify this as a Normal distribution with parameters that you should state explicitly.

(6)

(b) The random variables X_1, X_2, X_3 have the multivariate Normal distribution with expectation $\boldsymbol{\mu} = (\mu_1 \quad \mu_2 \quad \mu_3)^T$ and covariance matrix $\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & & \\ \sigma_{13} & & \boldsymbol{\Sigma}_{23} \end{pmatrix}$,

where $\boldsymbol{\Sigma}_{23}$ is a 2×2 sub-matrix. In general, the conditional distribution of X_1 given that $X_2 = x_2$, $X_3 = x_3$ is a Normal distribution with

$$E(X_1 \mid x_2, x_3) = \mu_1 + (\sigma_{12} \quad \sigma_{13})\boldsymbol{\Sigma}_{23}^{-1}\begin{pmatrix} x_2 - \mu_2 \\ x_3 - \mu_3 \end{pmatrix},$$

$$\text{Var}(X_1 \mid x_2, x_3) = \sigma_1^2 - (\sigma_{12} \quad \sigma_{13})\boldsymbol{\Sigma}_{23}^{-1}\begin{pmatrix} \sigma_{12} \\ \sigma_{13} \end{pmatrix}.$$

Obtain the parameters of the conditional distribution of X_1 given that $X_2 = x_2$, $X_3 = x_3$ in the special case where X_2 and X_3 are independent random variables. Find an expression for the multiple correlation of X_1 on both X_2 and X_3 in this case.

(10)

5. The continuous random variable X has the gamma distribution with parameters α and θ. The probability density function of X is given by

$$f(x) = \frac{\theta^\alpha x^{\alpha-1} e^{-\theta x}}{\Gamma(\alpha)}, \quad x > 0,$$

where $\alpha > 0$ and $\theta > 0$ and $\Gamma(.)$ denotes the gamma function. Furthermore, the continuous random variable Y has the gamma distribution with parameters $\beta > 0$ and θ. X and Y are independent random variables.

(i) Obtain the joint probability density function of U and V, where

$$U = \frac{X}{X+Y} \text{ and } V = X + Y .$$

State explicitly the region on which this joint probability density function is non-zero.

(11)

(ii) Explain how you know that U and V are independent. Show that V has a gamma distribution and identify its parameters. Write down the marginal distribution of U.

(5)

(iii) U has a beta distribution with parameters α and β. Show that $E(U) = \dfrac{\alpha}{\alpha+\beta}$.

Find $E(V)$.

(4)

6. The discrete random variables X_1, X_2, …, X_n ($n \geq 2$) are independent and each X_i has the Poisson distribution, $P(X_i = x_i) = \dfrac{e^{-\lambda_i} \lambda_i^{x_i}}{x_i\,!}$, for $\lambda_i > 0$ ($i = 1, 2, \ldots, n$).

(i) Show that X_i has moment generating function

$$M_i(t) = \exp\{\lambda_i (e^t - 1)\}\,.$$

Use this moment generating function to find $E(X_i)$ and $\mathrm{Var}(X_i)$.

(8)

(ii) Using moment generating functions, show that

$$S = X_1 + X_2 + \ldots + X_n$$

has the Poisson distribution with expected value $\lambda_1 + \lambda_2 + \ldots + \lambda_n$.

(4)

(iii) Let $S = s$, for some $s \geq 0$. What can you say about the possible values of (X_1, X_2, …, X_{n-1})? For (x_1, x_2, …, x_{n-1}) in this range, obtain the conditional probability mass function $p_{12\ldots(n-1)|S}(x_1, x_2, \ldots, x_{n-1} \mid S = s)$.

Express this function in the form

$$\frac{s!}{x_1! \ldots x_n!} p_1^{x_1} \cdots p_n^{x_n}$$

where $x_n = s - x_1 - \ldots - x_{n-1}$, stating explicitly the values of the parameters p_1, …, p_n.

(8)

7. A university café is considering introducing the following system for charging customers, in an attempt to reduce the need to keep a lot of change. Bills will be worked out as usual but the customer will always pay a whole number of pounds. If a bill comes to £A and x pence (for $x = 0, 1, \ldots, 99$), then the till will be programmed to randomly charge the customer either £A, with probability $1 - \frac{x}{100}$, or £$(A + 1)$, with probability $\frac{x}{100}$. For example, a customer whose bill is £7.14 will have probability 0.86 of being charged £7 and probability 0.14 of being charged £8.

(i) Suppose that X_1, the number of pence on a bill, is equally likely to be any of the values 0, 1, …, 99. Find $E(X_1)$ and $\text{Var}(X_1)$.

[Hint: you may use, without proof, the result that $\sum_{x=0}^{n} x^2 = \frac{1}{6}n(n+1)(2n+1)$.]

(4)

(ii) Let X_2 be the amount (in pence) that a customer gains (or loses) on a single transaction when this system is introduced. Find $E(X_2 \mid X_1)$ and $\text{Var}(X_2 \mid X_1)$. Hence find $E(X_2)$ and $\text{Var}(X_2)$, where

$$\text{Var}(X_2) = E\{\text{Var}(X_2 \mid X_1)\} + \text{Var}\{E(X_2 \mid X_1)\}.$$

(10)

(iii) A student who hears about this proposed system believes that she will eat in the café 100 times in the coming semester. Find the approximate probability that, in the course of the semester, the total amount she is charged in the café when this system is used will be no more than 100 pence (i.e. £1) different from the total amount of her bills. You may assume that the amounts gained or lost each time are independent random variables.

(6)

8. (a) Let X be any continuous random variable, and let $F(x)$ be its cumulative distribution function. Suppose that the continuous random variable U has a uniform distribution on the interval (0, 1). Define the new random variable Y by $Y = F^{-1}(U)$ (where $F^{-1}(.)$ is the inverse function of $F(.)$). By considering the cumulative distribution function of Y, or otherwise, show that Y has the same distribution as X.

(4)

(b) The following values are a random sample of numbers from a uniform distribution on the interval (0, 1):

0.149, 0.281, 0.534, 0.906.

Use these values to generate 4 random variates from each of the following distributions, carefully explaining the method you use in each case.

(i) Geometric: $P(X = x) = \left(\frac{1}{2}\right)^x, \quad x = 1, 2, \ldots$.

(6)

(ii) Pareto: $f(x) = \dfrac{24}{x^4}, \quad x > 2.$

(6)

(iii) Standard Normal: $f(x) = \dfrac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right), \quad -\infty < x < \infty.$

(4)

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