ROYAL STATISTICAL SOCIETY
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THE ROYAL STATISTICAL SOCIETY

2016 EXAMINATIONS − SOLUTIONS

GRADUATE DIPLOMA – MODULE 2

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Note that there are half-marks in some questions. Please round up any odd half-mark in a question in the total mark for that question.

1. (i) $E(X)=\int_\theta^1 x(1-\theta)^{-1}dx=(1-\theta)^{-1}[\frac{1}{2}x^2]_\theta^1=\frac{1-\theta^2}{2(1-\theta)}=\frac{1+\theta}{2}$ [1]

(or by a geometric argument)

Method of moments: set sample mean equal to population mean and solve for θ, [1]

so the estimator $\hat{\theta}$ satisfies $\bar{X}=\frac{1+\hat{\theta}}{2}$

i.e. $\hat{\theta}=2\bar{X}-1$ [1]

(ii) $$E(\bar{X})=E(\frac{1}{n}\sum_{i=1}^{n}X_i)=\frac{1}{n}[nE(X_i)]=\frac{1+\theta}{2}$$, so

$$E(\hat{\theta})=2E(\bar{X})-1=(1+\theta)-1=\theta$$

i.e. $\hat{\theta}$ is unbiased [1]

$$E(X^2)=\int_\theta^1 x^2(1-\theta)^{-1}=(1-\theta)^{-1}[\tfrac{x^3}{3}]_\theta^1=\frac{1-\theta^3}{3(1-\theta)}=\tfrac{1}{3}(1+\theta+\theta^2)$$ [1]

$$\text{var}(X)=\tfrac{1}{3}(1+\theta+\theta^2)-\left(\frac{1+\theta}{2}\right)^2=\tfrac{1}{12}(4+4\theta+4\theta^2-3-6\theta-3\theta^2)$$

$$=\tfrac{1}{12}(\theta^2-2\theta+1)=\frac{(1-\theta)^2}{12}$$ [1]

Hence $\text{var}(\hat{\theta})=4\,\text{var}(\bar{X})=\frac{4(1-\theta)^2}{12n}=\frac{(1-\theta)^2}{3n}$. [1]

(iii) Cramér-Rao lower bound: under certain regularity conditions (may be implicit) [1] the variance of any unbiased estimator of a parameter θ is bounded below by

$E\left[\left(\frac{\partial l}{\partial\theta}\right)^2\right]$ [1]

($-E\left[\left(\frac{\partial^2 l}{\partial\theta^2}\right)\right]$ also acceptable), where $l=\log(L(\theta))$ and $L(\theta)$ is the likelihood function.

One of the regularity conditions for the C-R lower bound is that the range of values for x should not depend on θ. This does not hold here, so the C-R is not applicable [1].

(iv) For $\theta < y < 1$, $P(Y > y) = P(X_1, X_2, ..., X_n > y) = \prod_{i=1}^{n} P(X_i > y)$ [1]

$$= \left(1 - \tfrac{(y-\theta)}{(1-\theta)}\right)^n = \left(\tfrac{1-y}{1-\theta}\right)^n \quad [1]$$

$$F(y) = 1 - \left(\tfrac{1-y}{1-\theta}\right)^n \text{ [1] so } f(y) = \frac{n(1-y)^{n-1}}{(1-\theta)^n} \quad [1]$$

(v) Mean square error of $\tilde{\theta}$ is

$E[(\tilde{\theta} - \theta)^2]$ [1] $= E[(1 - c(1-Y) - \theta)^2] = E[(1 - \theta - c(1-Y))^2]$

$$= (1-\theta)^2 - 2c(1-\theta)E[1-Y] + c^2 E[(1-Y)^2] \quad [1]$$

$$= (1-\theta)^2 - \frac{2c(1-\theta)n(1-\theta)}{(n+1)} + \frac{c^2 n(1-\theta)^2}{(n+2)} \quad [1]$$

[Candidates may possibly start from MSE = Variance + Bias^2. If they do and succeed in getting the correct final expression they should get 3 marks, with 1 or 2 marks for partially successful attempts.]

Differentiate w.r.t. c and equate to zero:

$$-\frac{2n(1-\theta)^2}{n+1} + \frac{2nc(1-\theta)^2}{n+2} = 0 \text{ [1] so c=} \frac{n+2}{n+1}. \text{ [1]}$$

The second derivative is positive, so this corresponds to a minimum. [1]

[An alternative argument not using calculus would be acceptable.]

2. (i) Suppose that $\hat{\theta}$ is the MLE of a parameter θ and $\phi = g(\theta)$ is a (1-1) function [1] of θ. Then $g(\hat{\theta})$ is the MLE of ϕ. [1]

(ii) The likelihood function is $L(\pi; y) = \binom{n}{y}\pi^y(1-\pi)^{n-y}$ [1]. Taking logs,

$l = \log(L) = const + y\, log(\pi) + (n-y)\log(1-\pi)$. [1]

Differentiating and equating to zero, $\frac{y}{\hat{\pi}} = \frac{(n-y)}{(1-\hat{\pi})}$, [1] so $\hat{\pi} = \frac{y}{n}$. [1].

$\frac{\partial^2 l}{\partial \pi^2} < 0$, this is the MLE [1].

(iii) $P(X_i < T) = \frac{1}{\theta}\int_0^T e^{-x/\theta}dx$ [1] $=1-e^{-T/\theta}$ [1] $=\phi$, say, and $P(X_i > T) = 1-\phi$.

We have a binomial [1] with n trials, probability of success ϕ, and *y* successes. [1]

Hence the MLE of ϕ is $\frac{y}{n}$ [1].

From part (i) $\hat{\phi} = 1-e^{-T/\hat{\theta}}$, [1] so $-T/\hat{\theta} = \log(1-\hat{\phi}) = \log(1-\frac{y}{n})$ [1] and

$\hat{\theta} = -T\left[\log\left(1-\frac{y}{n}\right)\right]^{-1}$ as required. [1]

(iv) The approximate distribution of $\hat{\theta}$ is $N(\theta, I_\theta^{-1})$, where $I_\theta = E\left[\left(\frac{\partial l}{\partial \theta}\right)^2\right] = E\left[-\frac{\partial^2 l}{\partial \theta^2}\right]$

(either expression OK) and *l* is the log likelihood. [1] mark is awarded for normality, [1] for the mean, [1] for either expression for the variance, and [1] for saying that *l* is the log likelihood. There is no need to use the notation I_θ. The reciprocal of one of the given expressions could be quoted directly as the variance of $\hat{\theta}$.

An approximate 95% confidence interval has end-points $\hat{\theta} \pm 1.96 I_\theta^{-1/2}$, [1]

3. (i) A statistic $T(X_1, X_2, ...X_n)$ is a function of $X_1, X_2, ...X_n$ but not of θ. [1]

It is sufficient for θ if the conditional distribution of $X_1, X_2, ...X_n$, given the value of T, does not depend on θ. [1] What this means is that T contains all the information about θ that is available in $X_1, X_2, ...X_n$. [1]

(ii) The likelihood is

$$L(\theta; \underline{x}) = \prod_{i=1}^{n} f(x_i; \theta) = \prod_{i=1}^{n} \exp\{A(\theta)B(x_i) + C(x_i) + D(\theta)\} \text{ [1]}$$

$$= \exp\{A(\theta)\sum_i B(x_i) + \sum_i C(x_i) + nD(\theta)\} \text{ [1]}$$

[This final expression is not strictly needed in answering (ii), but is needed in (iii). It should be awarded a mark whether it appears in the answer to (ii) or to (iii).]

(iii) Write $K_1(T; \theta) = \exp\{A(\theta)\sum_i B(x_i) + nD(\theta)\}$ [1] and

$K_2(\underline{x}) = \exp\{\sum_i C(x_i)\}$. [1] Then $L(\theta; \underline{x}) = K_1(T; \theta)K_2(\underline{x})$, where

$T = \sum_i B(x_i)$ is a sufficient statistic for θ. [1]

(iv) The likelihood function is

$$L(\theta; \underline{x}) = \prod_i \frac{x_i}{\theta} \exp\left(\frac{-x_i^2}{2\theta}\right)$$

$$= \exp\{A(\theta)\sum_i B(x_i) + \sum_i C(x_i) + nD(\theta)\} \text{ [1]}$$

where $A(\theta) = \frac{-1}{2\theta}$, $B(x_i) = \sum_i x_i^2$, $C(x_i) = \log(x_i)$ and $D(\theta) = -\log(\theta)$. [1]

Hence the distribution is a member of the one-parameter exponential family [1] and $B(x_i) = \sum_i x_i^2$ is a sufficient statistic for θ. [1]

(v) A prior distribution represents knowledge about the probability distribution of an unknown parameter θ before any data are considered. [1]

Combining the prior distribution with the likelihood function gives the posterior distribution. [1] A conjugate prior distribution is such that the resulting posterior distribution belongs to the same family as the prior distribution. [1]

The likelihood function for the Rayleigh distribution can be written as

$$L(\theta;\underline{x}) = \frac{\prod_i x_i}{\theta^n} \exp\left(\frac{-\sum_i x_i^2}{2\theta} \right) \text{ [1]}$$

When multiplied by a prior distribution of the given form, we get a posterior distribution proportional to $\theta^{-\beta_1} \exp(-\beta_2/\theta)$, where $\beta_1 = \alpha_1 + n$ [1] and $\beta_2 = \alpha_2 + \frac{1}{2}\sum_i x_i^2$. [1]

This is of the same form as the prior distribution with ‘updated’ parameter values [1] so the family of prior distributions given is indeed conjugate. [1]

4 (i) Suppose that we are testing a null hypothesis H0: $\underline{\theta} \in \omega$ against the alternative H1: $\underline{\theta} \in \Omega - \omega$ [1]. The test statistic for a generalised likelihood ratio test of H0 vs. H1 is

$$\lambda = \underset{\underline{\theta} \in \omega}{Max}\left\{L(\underline{\theta};\underline{x})\right\} \Big/ \underset{\underline{\theta} \in \Omega}{Max}\left\{L(\underline{\theta};\underline{x})\right\} \text{ , [1]}$$

where $L(\underline{\theta};\underline{x})$ is the likelihood function. [0.5]

Reject H0 for small values of λ. [0.5]

(ii) For large samples $-2\log\lambda \sim \chi_d^2$ [0.5] where *d* is the difference between the number of freely varying independent parameters in Ω and in ω. [0.5]

For H0: $\theta = \theta_0$ vs. H_1: $\theta \neq \theta_0$, *d*=1 [1] and $\lambda = L(\theta_0;\underline{x})/L(\hat{\theta};\underline{x})$, where $\hat{\theta}$ is the maximum likelihood estimator for θ. [1]

$-2\log\lambda = 2[l(\hat{\theta};\underline{x}) - l(\theta_0;\underline{x})]$, where $l(\theta;\underline{x})$ is the log-likelihood [1]. Using the approximation $\Pr[2(l(\hat{\theta};\underline{x}) - l(\theta_0;\underline{x})) \leq \chi_{1;\alpha}^2] = 1-\alpha$, where $\chi_{1;\alpha}^2$ is the upper α critical point of χ_1^2, [1] a confidence interval for θ with confidence coefficient $(1-\alpha)$ is given by those values of θ for which

$l(\theta;\underline{x}) \geq l(\hat{\theta};\underline{x}) - \frac{1}{2}\chi_{1;\alpha}^2$ [1]

(iii) For a random sample of n observations $x_1, x_2, ..., x_n$ from a Poisson distribution with mean μ the likelihood function is $\prod_{i=1}^{n}\frac{\mu^{x_i}e^{-\mu}}{x_i!} = \frac{\mu^{\sum_{i=1}^{n}x_i}e^{-n\mu}}{\prod_{i=1}^{n}x_i}$. [1]

The log likelihood is $l(\mu;\underline{x}) = const + \sum_i x_i \log\mu - n\mu$. [1]

Differentiating, equating the derivative to zero, [1] and checking the second derivative to confirm a *max*imum,[0.5] gives the MLE as $\hat{\mu} = \sum_i x_i \Big/ n = \bar{x}$ [1].

The generalised likelihood ratio test statistic for testing H0: $\mu = 5$ against a two-sided alternative is $-2\log\lambda = 2[l(\hat{\mu};\underline{x}) - l(5;\underline{x})]$. [1]

$\hat{\mu} = \bar{x} = 27/9 = 3,$ [0.5]

$-2\log\lambda = 2[27(\log 3 - \log 5) - 9(3-5)] = 2[-13.79 + 18] = 8.42$. [1]

This is well above the 5% critical value for χ_1^2 (3.84) [0.5], so the null hypothesis is rejected. So there is evidence that the accident rate has changed [0.5] (actually decreased).

(iv) From part (ii), the endpoints of the interval satisfy the equation $l(\mu;\underline{x}) - l(3;\underline{x}) = -{}^{3.84}\!/_{2}$ [1] giving $(27\log\mu - 9\mu) - (27\log 3 - 9\text{x}3) = -1.92$, [1] which simplifies to $3\log\mu - \mu = 0.0825$ as required. [1]

5 (a) Denote the vector of values $x_1, x_2, ..., x_n$ by $\underline{x}$. Find a function $g(\underline{x};\theta)$ of $\underline{x}$ and θ which is monotonic in θ [1] and whose probability distribution is known and does not depend on θ. [1] This is called a pivotal quantity. [1] Make a probability statement regarding $g(\underline{x};\theta)$ i.e $\Pr[g_1 \le g(\underline{x};\theta) \le g_2] = 1-\alpha$ where α is fixed (typically 0.05 or 0.01) [1] and g_1, g_2 do not depend on θ. Now manipulate the inequalities in the probability statement to get θ 'in the middle' [1] i.e $\Pr[\theta_1(\underline{X}) \le \theta \le \theta_2(\underline{X})] = 1-\alpha$. The interval $(\theta_1(\underline{X}), \theta_2(\underline{X}))$ is a $100(1-\alpha)\%$ confidence interval for θ. [1]

[If the monotonicity condition is not mentioned, full marks can still be achieved provided that it is mentioned that the resulting confidence set need not be a single interval.]

(b) In the Bayesian framework, θ is considered to be a random variable [1] and so has a probability distribution. A prior distribution is specified, [1] before taking any data into account. The likelihood function represents the information about θ contained in the data $\underline{x}$. [1] The posterior distribution for θ is obtained by taking the product of the prior distribution and the likelihood function and normalising so that it integrates to1. [1] A $100(1-\alpha)\%$ credible interval for θ is given by (θ_1, θ_2) such that $\Pr[\theta_1 \le \theta \le \theta_2] = 1-\alpha$ according to the posterior distribution. [1]

(c) Suppose that an estimate $\hat{\theta}(\underline{x})$ based on $\underline{x} = (x_1, x_2, ..., x_n)$ is available for θ. [1] Take a sample of size n *with replacement* from $x_1, x_2, ... x_n$ - call it $\underline{x}_1^*$ - and calculate $\hat{\theta}(\underline{x}_1^*)$. [1] Repeat the sampling with replacement (a large number) B times, to give B estimates of θ. [1] Arrange these B estimates in ascending order to give $\hat{\theta}_{[1]}^* \le \hat{\theta}_{[2]}^* \le ... \le \hat{\theta}_{[B]}^*$. [1] Let $B\alpha/2 = m$ (ideally choose B so that this is an integer). Then a $100(1-\alpha)\%$ bootstrap percentile confidence interval for θ is $(\hat{\theta}_{[m]}^*, \theta_{[B-m+1]}^*)$. [1]

Interpretation of frequentist interval: the parameter θ is fixed but unknown – the interval is random. [1] In many repetitions of finding $100(1-\alpha)\%$ confidence intervals, the intervals with contain the true value of the parameter $100(1-\alpha)\%$ of the time in the long run, but there no information on whether any individual interval will do so. [1]

Interpretation of Bayesian interval: θ is considered to be a random variable so the interval is an interval between two quantiles of its posterior distribution. [1] So in this case the end-points of the interval are fixed (not random – unlike the frequentist interval). [1]

6. (i) Let $F_0(x)$ be the c.d.f. of the null distribution. [1]

If the random sample is ordered as $X_{[1]} \le X_{[2]} \le ... \le X_{[n]}$ define

$$F_n(x) = \begin{cases} 0 & \text{if } 0 < x < X_{[1]} \\ \dfrac{k}{n} & \text{if } X_{[k]} \le x < X_{[k+1]},\ k = 1, 2, ..., n-1 \\ 1 & \text{if } x \ge X_{[n]} \end{cases} \qquad [1]$$

Then the test statistic for the one-sample Kolmogorov-Smirnov test is
$\underset{x}{Sup} \, | F_0(x) - F_n(x) |$ [1]

[The null hypothesis will be rejected for large values of the test statistic. If this is stated here by candidates, but not explicitly mentioned in their solution to (ii), 1 additional mark should be awarded here.]

(ii) Recognise that $1 - e^{-\frac{1}{2}x}$ is the c.d.f. for an exponential distribution with mean 2 [1] and that $| F_0(x) - F_n(x) |$ must reach its maximum value at or immediately below one of the observed values of x. [1] It rises to 0.167 just below $x_{[1]}$, then drops to 0.033; it rises to 0.238 just below $x_{[2]}$, then drops to 0.038; rises to 0.359 just below $x_{[3]}$, then drops to 0.159; rises to 0.293 just below $x_{[4]}$, then drops to 0.093; rises to 0.099 just below $x_{[5]}$ before jumping to 0.101, and finally decreasing to zero. [2 marks if all these calculations are correct, 1 mark if method is clearly known but calculations are wrong.]

So the test statistic has value 0.359. [1] Large values of the test statistic lead to rejection of the null hypothesis [1] and 0.359 is well short of the given critical values [1]. Thus there is insufficient evidence to reject H_0 [1]

(iii) The t-test is a test for the mean [1] whereas the sign test is a test for the median [1]

The distribution of the test statistic for the t-test assumes approximate normality for the data which is clearly not the case here. [1] The sign test is non-parametric and does not depend on the distribution of the data and so can be used here. [1]

(iv) If the distribution is exponential with mean 2, then its median m is found by solving $0.5 = 1 - e^{-\frac{1}{2}m}$ [1] so $\text{m} = 2\log 2 = 1.386$ [1]. Hence test $H_0 : m = 1.386$ against $H_1 : m \ne 1.386$ [1]

The test statistic S is the number of observations greater than 1.386, which is 3. [1]   Since this is right in the centre of the distribution of S [Bin(5, 0.5)], there is clearly no evidence against the null hypothesis. [1]

[Although I'm not expecting it, I would award full marks if a candidate calculated Pr(X>2) = 0.368, and used a test statistic equal to the number of observations exceeding 2, with null distribution Bin(5, 0.368)].

7. (a) If p is the probability of H₀ then $\frac{p}{(1-p)}$ is the odds of the H₀. [1]

Given two simple hypotheses H_0, (null) H_1 (alternative) the Bayes factor is the likelihood under H_0 divided by the likelihood under H_1 [1] [The reciprocal of this would also be acceptable, although the wording of the question should steer candidates towards this definition].

In Bayesian inference for a parameter θ, the posterior distribution of θ is given by $q(\theta \mid \underline{x}) = \frac{L(\theta;\underline{x})\, p(\theta)}{h(\underline{x})}$, where $p(\theta)$ is the prior distribution, $L(\theta;\underline{x})$ is the likelihood function, and $h(\underline{x})$ is the marginal distribution of the data. [1]

Suppose the simple hypotheses are $H_0 : \theta = \theta_0 ; H_1 : \theta = \theta_1$. Then

$$\text{Posterior odds} = \frac{q(\theta_0 \mid \underline{x})}{q(\theta_1 \mid \underline{x})} = \frac{L(\theta_0;\underline{x})\, p(\theta_0)}{h(\underline{x})} \bigg/ \frac{L(\theta_1;\underline{x})\, p(\theta_1)}{h(\underline{x})} \quad [1]$$

$$= \frac{p(\theta_0)}{p(\theta_1)} \frac{L(\theta_0;\underline{x})}{L(\theta_1;\underline{x})} = \text{prior odds x Bayes factor, as required.}[1]$$

(b) (i) $\Pr(X_i = x_i) = \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$ [1]

so $L(\lambda;\underline{x}) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \frac{e^{-n\lambda}\lambda^{\sum_i x_i}}{\prod_i x_i!}$ [1]

The Bayes factor is this likelihood evaluated at $\lambda = 5$ divided by the likelihood evaluated at $\lambda = 10$. The factorial terms cancel leaving

$e^{-5n} 5^{\sum_i x_i} \Big/ e^{-10n} 10^{\sum_i x_i}$ [1] $= e^{5n} (0.5)^{\sum_i x_i}$ [1]

(ii) Posterior odds will be greater than prior odds if the Bayes factor exceeds 1. [1]

This occurs if $e^{5n}(0.5)^{\sum_i x_i} > 1$ or $(0.5)^{\sum_i x_i} > e^{-5n}$

$\sum_i x_i \log(0.5) > -5n;$ [1] $\sum_i x_i < \frac{-5n}{\log(0.5)};$ [1] $\bar{x} < \frac{5}{\log(2)}$, as required [1]

(iii) Prior distribution has p.d.f. $0.2e^{-0.2\lambda}$, $\lambda > 0$ [0.5] and the likelihood function is $\frac{e^{-n\lambda}\lambda^{\sum_i x_i}}{\prod_i x_i!}$ [0.5].

The likelihood for a composite hypothesis such as H_1 is obtained by integrating the product of this likelihood function and the prior distribution over values of λ contained in the hypothesis, [1] so

$$L(H_1;\underline{x}) = \int_0^\infty \frac{e^{-n\lambda}\lambda^{\sum_i x_i} 0.2e^{-0.2\lambda}}{\prod_i x_i!} d\lambda = \frac{0.2}{\prod_i x_i!}\int_0^\infty \lambda^{\sum_i x_i} e^{-\lambda(n+0.2)} d\lambda \text{ [1]}$$

$$= \frac{0.2}{\prod_i x_i!(n+0.2)^{(\sum_i x_i)+1}} \int_0^\infty (n+0.2)^{(\sum_i x_i)+1} \lambda^{(\sum_i x_i)+1-1} e^{-\lambda(n+0.2)} d\lambda \text{ [1]}$$

The integral is in the form of the gamma function given in the hint with

$\nu = n+0.2$ and $\alpha = (\sum_i x_i)+1$ [1] so $L(H_1;\underline{x}) = \frac{0.2\Gamma(\sum_i x_i+1)}{\prod_i x_i!(n+0.2)^{(\sum_i x_i)+1}}$ [1] and

the Bayes factor is

$$\frac{L(H_0;\underline{x})}{L(H_1;\underline{x})} = \frac{\dfrac{e^{-5n}5^{\sum_i x_i}}{\prod_i x_i}}{\dfrac{0.2\Gamma(\sum_i x_i+1)}{\prod_i x_i!(n+0.2)^{(\sum_i x_i)+1}}} = \frac{e^{-5n}(5n+1)^{\sum_i x_i+1}}{\Gamma(\sum_i x_i+1)} \text{ as required. [1]}$$

8. (i) A strategy d_i is inadmissible if there is another strategy d_j for which $U(d_i, \theta_k) \le U(d_j, \theta_k)$ for all k, where $U(d_i, \theta_k)$ is the utility of strategy when state of nature holds, [1] with at least one inequality strict [1].

Clearly d_1 is inadmissible (compare with any other strategy), as is d_2 (compare with d_3, d_4). [1].

A maximin strategy is one for which the minimum utility is maximised. [1] The minimum utility is 1.0, 0.5, -4.5 for d_3, d_4, d_5 respectively (inadmissible strategies don't need to be considered, but candidates will not be penalised if they do so). So d_3 is maximin. [1]

(ii) The Bayes strategy is the one which maximises expected utility, [1] where expectation is taken with respect to the prior distribution of the states of nature. [1]

Prior probabilities of $\theta_1, \theta_2, \theta_3, \theta_4$ are 0.4, 0.4, 0.1, 0.1 respectively [1], and the expected utilities for the three admissible strategies are:

d_3: 0.4x1 + 0.4x1 + 0.1x2 + 0.1x2 = 1.2;

d_4: 0.4x0.5 + 0.4x0.5 + 0.1x2.5 + 0.1x 2.5 = 0.9;

d_5: -0.4x4.5 + 0.4x1 - 0.1x1.5 + 0.1x4 = -1.15. [1]

So d_3 is the Bayes strategy. [1]

(iii) Expected utilities are

d_3: $\pi_1 + 2(1-\pi_1) = 2 - \pi_1$

d_4: $0.5\pi_1 + 2.5(1-\pi_1) = 2.5 - 2\pi_1$

d_5: $-1.75\pi_1 + 1.25(1-\pi_1) = 1.25 - 3\pi_1$ [1]

It is fairly clear that the expected utility for d_5 is smaller than that for d_3 for any value of π_1 between 0 and 1, so d_5 is never Bayes. [1]

d_3 is better than d_4 if $2 - \pi_1 > 2.5 - 2\pi_1$ i.e. $\pi_1 > 0.5.$ So d_3 is Bayes when π_1 >0.5 and d_4 is Bayes for $\pi_1 < 0.5$ [1] (Both are equally good at $\pi_1 = 0.5$).

(iv) Posterior probability for θ_i given advice of small demand, is the product of the prior probability and the probability of small demand, given such advice, divided by the probability of such advice. [1]

For θ_1 this is $\frac{0.4\text{x}0.9}{1-\phi} = \frac{0.36}{1-\phi}$. For $\theta_2, \theta_3, \theta_4$, the corresponding values are $\frac{0.04}{1-\phi}, \frac{0.09}{1-\phi}, \frac{0.01}{1-\phi}$. [1]

For d_3 the expected utility is now $\frac{(0.36+0.04)+2(0.09+0.01)}{1-\phi} = \frac{0.6}{1-\phi}$ [1]

Candidates could similarly calculate expected utilities for d_4, d_5, but it would be equally acceptable to say that looking at the table of utilities and the dominance of the prior probability for θ_1, that it is clear that the expected utilities for d_4, d_5 will be less than that for d_3 [1] so d_3 is Bayes when the advice is for small demand. [1]

The expected utility if the advice is used is (Expected utility when advice is large)x(Probability advice is 'large') + (Expected utility when advice is 'small')x(Probability advice is 'small') $= \frac{0.6625\phi}{\phi} + \frac{0.6(1-\phi)}{(1-\phi)} = 1.2625$. [1]

In the absence of advice, part (ii) showed that the optimal strategy was d_3 with expected utility 1.2. So the advice increases expected utility by 0.0625 i.e £62500 compared to a fee of £50000, so the gain from using the consultancy firm's advice is £12500. [1]