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The solutions are intended as learning aids and should not be seen as "model answers".

Users of the solutions should always be aware that in many cases there are valid alternative methods. Also, in the many cases where discussion is called for, there may be other valid points that could be made.

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GD3 (2016) Solutions

None of the states communicate so each forms a class. 1, 2 and 3 are transient while 4 is absorbing so is recurrent. Someone with the disease will end up in state 4 with probability 1, i.e. permanent disability.

\[ P^{(2)} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{16} & \frac{7}{16} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

\[ P^{(3)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{pmatrix} \]

All four states now communicate so form a single recurrent class.

\[ \Pi = \Pi^* \quad \text{subject to } \sum_{i=1}^{4} \Pi_i = 1 \]

\[ \begin{align*}
\Pi_1 &= \frac{1}{2} \Pi_1 + \frac{1}{2} \Pi_4 \quad \Rightarrow \quad \Pi_1 = \frac{1}{3} \\
\Pi_2 &= \frac{1}{4} \Pi_1 + \frac{1}{2} \Pi_2 \quad \Rightarrow \quad \Pi_2 = \frac{1}{2} \\
\Pi_3 &= \frac{1}{4} \Pi_2 + \frac{1}{2} \Pi_3 \quad \Rightarrow \quad \Pi_3 = \frac{1}{2} \\
(\Pi_4 &= \frac{1}{4} \Pi_1 + \frac{1}{4} \Pi_2 + \frac{1}{2} \Pi_3 + \frac{1}{2} \Pi_4) \]

\[ \sum_{i=1}^{4} \Pi_i = 1 \quad \Rightarrow \quad (1 + \frac{1}{2} + \frac{1}{2} + 1) \Pi_1 = 1 \quad \Rightarrow \quad \Pi_1 = \frac{4}{11} \]

\[ \begin{align*}
\Pi_2 &= \frac{7}{11} \Pi_1 \\
\Pi_3 &= \Pi_1 \\
\Pi_4 &= \frac{4}{11} \\
\end{align*} \]

. P(Receiving treatment) = \Pi_4 = \frac{4}{11} \quad \text{regardless of the initial state.}

\[ E[\text{Cost}] = 0 + \frac{7}{11}c + \frac{1}{11} \times 2c + \frac{4}{11} \times 8c = \frac{36}{11}c \]
2. (i) Since $X_1$ consists of a single family $Z$, $G_1(s) = \mathbb{E}[s^{X_1}] = G(s)$.

(ii) $X_{ni}$ is the total population of $i$ (sub) branching processes with $(n-1)$ generations from one individual. Hence $\mathbb{E}[s^{X_{ni}}] = G_{n-1}(s)$.

(iii) $G_n(s) = \mathbb{E}[s^{X_n}] = \mathbb{E}\left[\mathbb{E}[s^{X_n} | X_i]\right]$.
Now if $X_1 = x$, $X_n = \sum_{i=1}^{x} X_{ni}$. so $\mathbb{E}[s^{X_n} | X_i = x] = \prod_{i=1}^{x} \mathbb{E}[s^{X_{ni}}] = G_{n-1}(s)^x$, by independence.
\[ \mathbb{E}[s^{X_n} | X_i = x] = G_{n-1}(s)^x, \quad \text{by (ii)}, \]
and $G_n(s) = \mathbb{E}[G_{n-1}(s)^{X_i}] = G(G_{n-1}(s))$ since $\mathbb{E}[s^{X_i}] = G(s)$, by (i).

(iv) $X_n = 0 \Rightarrow X_{n+1} = 0$ so $\Pi_n \leq \Pi_{n+1}$ and $\Pi_n$ is increasing.
$\Pi_n = G_n(0) = G(G_{n-1}(0)) = G(\Pi_{n-1})$.

(v) $\Pi = \lim_{n \to \infty} \Pi_n = \lim_{n \to \infty} G(\Pi_{n-1}) = G(\Pi)$.
Since $G(s) = \sum_{s=0}^{\infty} s^k p_k$ is continuous, $G(1) = \sum_{s=0}^{\infty} p_s = 1$ so $s=1$ is a root.

(vi) $G(s) = 0.1 + 0.2s + 0.3s^2 + 0.4s^3$
so solve $0.1 + 0.2s + 0.3s^2 + 0.4s^3 = s$ or $4s^3 + 3s^2 - 8s + 1 = 0$.
$s=1$ is a root so
\[(s-1)(4s^2 + 7s - 1) = 0\]
and the remaining roots are
\[-7 \pm \sqrt{49 + 16} \over 8 \]
\[i.e. \quad -1.88 \text{ or } 0.13\]
Hence $\Pi = 0.13$ as it is the smallest non-negative root.
Let $N(t) = N^x$ in system at time $t$. Then

$$
P \in N(t+\delta t) = i+1 \mid N(t) = i \xrightarrow{d} \Delta i \delta t + o(\delta t)
$$

$$
P \in N(t+\delta t) = i-1 \mid N(t) = i \xrightarrow{d} \beta_i \delta t + o(\delta t)
$$

Since $\sum_{i=0}^{\infty} \Pi_i = 1$, $\Pi_0 = \sum_{n=1}^{\infty} \frac{x_0 x_1 \ldots x_{n-1}}{\rho_1 \rho_2 \ldots \rho_n}$ provided it converges.

(i) (a) \[ \alpha_i = \lambda \frac{1}{i+1}, i = 0, 1, 2, \ldots \quad \beta_i = \frac{\mu}{i+1}, i = 1, 2, \ldots \]

\[ \Pi_n = \frac{x_0 x_1 \ldots x_{n-1}}{\rho_1 \rho_2 \ldots \rho_n} = \frac{x^n}{n! \rho n} = \frac{x^n}{n! \rho^n} = (\frac{x}{\rho})^n \]

where $\rho = \frac{\lambda}{\mu}$

\[ \Pi_0^{-1} = 1 + \sum_{n=1}^{\infty} (\frac{\rho}{1-\rho})^n = \sum_{n=1}^{\infty} (\frac{\rho}{1-\rho})^{n-1} = \frac{1}{1-\rho} \]

since $\rho < 1$.

Server is busy a proportion $1 - \Pi_0 = 1 - (1-\rho)^2 = \rho (2-\rho)$

(b) \[ \Pi_n = (1-\rho)^2 (n+1) \rho^n, \text{ for } n \geq 0 \]

\[ \text{Pgf is } P(2) = \sum_{n=0}^{\infty} z^n (1-\rho)^2 (n+1) \rho^n = (1-\rho)^2 \sum_{j=1}^{\infty} j (\rho z)^{j-1} = \frac{1-\rho^2}{(1-\rho z)^2} \]

\[ P'(2) = (1-\rho)^2 2 \rho (1-\rho z)^{-3} \Rightarrow E[N] = P'(1) = \frac{2 \rho}{1-\rho} \]

\[ P''(2) = (1-\rho)^2 6 \rho^2 (1-\rho z)^{-4} \Rightarrow E[N(N-1)] = P''(1) = \frac{6 \rho^2}{(1-\rho)^2} \]

\[ \text{Var}[N] = \frac{6 \rho^2 + 2 \rho - (2 \rho)^2}{(1-\rho)^2} = \frac{2 \rho}{(1-\rho)^2} \]

(c) \[ E[N^0 \text{ lost} \mid n \text{ in system}] = \lambda x [1 - \frac{1}{n+1}] = \lambda \frac{1}{n+1} \]

\[ E[N^0 \text{ lost}] = \sum_{n=0}^{\infty} \lambda n \frac{1}{n+1} \]

\[ = \lambda (1-\rho)^2 \rho \sum_{j=1}^{\infty} j \rho^{j-1} = \lambda \rho \frac{1}{(1-\rho)^2} \]
Let \( N(t) = N^2 \) of particles in \([0, t] \).
\[
P(N(t)=n | N(t)=n) = \frac{1 - \lambda(t)St + o(St)}{1 - \lambda(t)St + o(St)} \quad (n = 0, 1, 2, \ldots)
\]
\[
P(N(t+St)=n+1 | N(t)=n) = \lambda(t)St + o(St)
\]
\[
P(N(t+St) > n+2 | N(t)=n) = o(St)
\]

I. \( P_n(t+St) = \left(1 - \lambda(t)St + o(St)\right) P_n(t) + \left\{ \lambda(t)St + o(St) \right\} P_{n-1}(t) + o(St) \)

II. \( P_n(t+St) - P_n(t) = \lambda(t) P_{n-1}(t) + o(St) \)

\[
\frac{d}{dt} P_n(t) = -\lambda(t) P_{n-1}(t), \text{ letting } St \rightarrow 0
\]

Also \( \frac{d P_0(t)}{dt} = \lim_{St \rightarrow 0} \frac{P_0(t+St) - P_0(t) + o(St)}{St} = -\lambda(t) P_0(t) \)

(i) Multiplying by \( S^n \) and summing \( \sum_{n=0}^{\infty} \), we get:
\[
\sum_{n=0}^{\infty} \frac{d P_n(t)}{dt} S^n = -\lambda(t) \sum_{n=0}^{\infty} P_n(t) S^n + \lambda(t) \sum_{n=0}^{\infty} P_{n-1}(t) S^n
\]
\[
= -\lambda(t) G(s, t) + \lambda(t) \sum_{n=0}^{\infty} P_n(t) S^n
\]
\[
= -\lambda(t) G(s, t) + \lambda(t) S \cdot G(s, t)
\]

\[
\frac{dG(s, t)}{dt} = \lambda(t) (S-1) G(s, t), \text{ as required}
\]

Since \( P_0(0) = 1 \), \( G(s, 0) = \sum_{n=0}^{\infty} P_n(0) = s^0 = 1 \)

(iii)
\[
\int_0^t \frac{G(s, u)}{G(s, u)} du = \int_0^t \lambda(u) du (S-1) = \Lambda(t) (S-1)
\]
\[
\ln G(s, t) = \Lambda(t) (S-1) + \text{constant}
\]
\[
G(s, t) = e^{\Lambda(t) (S-1)}
\]
To satisfy \( G(s, 0) = 1 \), \( C(s) = 1 \) since \( \Lambda(0) = 0 \), which yields \( G(s, t) = e^{\Lambda(t) (S-1)} \).

(iv) Since \( G(s, t) = e^{\Lambda(t) S} \) and
\[e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots\]
\[G(s, t) = e^{-\Lambda(t) S} \left( 1 + \Lambda(t) S + \cdots + \frac{\Lambda(t)^n S^n}{n!} + \cdots \right)\]

Comparing with \( G(s, t) = \sum_{n=0}^{\infty} P_n(t) S^n \), we get
\[P_n(t) = e^{-\Lambda(t)} \frac{\Lambda(t)^n}{n!} \quad \text{for } n = 0, 1, 2, \ldots, \]
\[\text{i.e., Poisson (} \Lambda(t) \text{)}\]
(i) \( L_t = \alpha y_t + (1-\alpha) L_{t-1} \)

(ii) \( L_t = L_{t-1} + \alpha (y_t - L_{t-1}) \)
\[ = L_{t-1} + \alpha e_t \] since \( L_{t-1} = Y_{t-1} \)

(iii) \( L_t = \alpha y_t + (1-\alpha) (\alpha y_{t-1} + (1-\alpha) L_{t-2}) \)
\[ = \alpha y_t + (1-\alpha) \alpha y_{t-1} + (1-\alpha)^2 (\alpha y_{t-2} + (1-\alpha) L_{t-3}) \]
\[ = \alpha y_t + (1-\alpha) y_{t-1} + \ldots + (1-\alpha)^{t-2} (\alpha y_2 + (1-\alpha) L_1) \]
\[ = \alpha \sum_{i=0}^{t-2} (1-\alpha)^i y_{t-i} + (1-\alpha)^{t-1} y_1 \] since \( L_1 = Y_1 \).

(iv) \( \alpha = 0.3 \) 

<table>
<thead>
<tr>
<th>( )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<tbody>
<tr>
<td>( Y_t = )</td>
<td>13</td>
<td>15</td>
<td>12</td>
<td>17</td>
</tr>
<tr>
<td>( y_t = )</td>
<td>2</td>
<td>1.6</td>
<td>3.88</td>
<td>14.28</td>
</tr>
<tr>
<td>( y_t - L_{t-1} = e_t )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( L_t = L_{t-1} + 0.3 e_t )</td>
<td>13</td>
<td>13 + 0.3 ( \times ) 2 = 13.6</td>
<td>13.6 - 0.3 ( \times ) 1.6 = 13 ( \times ) 12</td>
<td>13 ( \times ) 12 + 0.3 ( \times ) 3.88 = 14.28</td>
</tr>
</tbody>
</table>

(b) \( Y_t = X_t - X_{t-1} = (W_t + A_t) - (W_{t-1} + A_{t-1}) \)
\[ = W_{t-1} + \Delta t + A_t - W_{t-1} - A_{t-1} \]
\[ = \Delta t + A_t - A_{t-1} \]

Hence \( \text{Var}(Y_t) = \delta^2 \Delta A^2 + 2 \Delta A^2 \) since \( \text{Cov}(A_t, A_{t-1}) = 0 \) and \( \text{Cov}(Y_t, Y_{t-1}) = \text{Cov}(\Delta t + A_t - A_{t-1}, \Delta t + A_{t-1} - A_{t-2}) \)
\[ = \text{Var}(A_{t-1}) = -2 \Delta A^2 \) since all other terms = 0.

\[ \rho_Y(1) = \frac{\text{Cov}(Y_t, Y_{t-1})}{\text{Var}(Y_t)} = \frac{-2 \Delta A^2}{2 \Delta A^2 + 2 \Delta A^2} \]

Also \( \text{Cov}(Y_t, Y_{t-k}) = \text{Cov}(\Delta t + A_t + A_{t-1}, \Delta t + A_{t-1} + A_{t-2} + A_{t-2} - 1) \)
\[ = 0 \]
so \( \rho_Y(k) = 0 \) for \( k \geq 2 \)

\( Y_t \) will have the acf of an MA(1) so \( Y_t \) will be ARIMA(0, 1, 1) i.e \( p = 0, d = 1, q = 1 \).
The model is \( MA(2) \) or \( ARMA(0,2) \): \( p = 0, q = 2 \).

(a) \( X_t \) is stationary since all pure \( MA(q) \) are stationary.
\[
1 + 0.4^2 + (-0.45)^2 < \infty
\]
(or the AR polynomial \( \phi(B) = 1 \) has no roots, hence none lie outside the unit circle.

(b) Roots of \( \Theta(B) = 1 + 0.4B - 0.45B = (1 + 0.9B)(1 - 0.5B) \).
Both roots \( B_1 = -0.9 \) and \( B_2 = 2 \) lie outside \( |B| = 1 \).

\( \Theta(B) \) is invertible.
\[
E[X_t] = 10 + E[A_{t1} + 0.4E[A_{t-1} - 0.45E[A_{t-2}]] = 10
\]
since \( E[A_{t-i}] = 0 \) for all \( i \).
\[
Var[X_t] = Var[A_{t1}] + 0.4Var[A_{t-1}] + (-0.45)^2 Var[A_{t-2}] = 1.3625 \sigma^2
\]
since \( Cov(A_{t-i}, A_{t-j}) = 0 \) \( i \neq j \).

\[
X_t(k) = E[X_{t+k} \mid X_t, X_{t-1}, \ldots] = 10 + E[A_{t+k} + 0.4A_{t+k-1} - 0.45A_{t+k-2} \mid X_t, X_{t-1}, \ldots] = 10, \text{ for } k \geq 3 \text{ since } E[A_{t+j} \mid X_t, X_{t-1}, \ldots] = 0
\]
for \( j > 3 \) due to independence of \( A_{t+j} \) from past \( X_t \).

\[
X_t(1) = E[X_{t+1} \mid X_t, X_{t-1}, \ldots] = 10 + E[A_{t+1} + 0.4A_t - 0.45A_{t-1} \mid X_t, X_{t-1}, \ldots] = 10 + 0 + 0.4 \alpha_t - 0.45 \alpha_{t-1}
\]
where \( \alpha_t = E[A_t \mid X_t, X_{t-1}, \ldots] \approx X_t - X_{t-1}(1) \)
and \( \alpha_{t-1} = E[A_{t-1} \mid X_t, X_{t-1}, \ldots] \approx X_{t-1} - X_{t-2}(1) \)
are 1-step ahead forecast errors.

Similarly \( X_t(2) = 10 + E[A_{t+2} + 0.4A_{t+1} - 0.45A_t \mid X_t, X_{t-1}, \ldots] = 10 + 0 + 0.4 \times 0 - 0.45 \alpha_t = 10 - 0.45 \alpha_t \)

\[
V(1) = Var[A_{t+1}] = \sigma^2, \quad V(2) = Var[A_{t+2} + 0.4A_{t+1}] = 1.16 \sigma^2
\]
\[
V(k) = Var[A_{t+k} + 0.4A_{t+k-1} - 0.45A_{t+k-2}] = 1.3625 \sigma^2
\]
for \( k \geq 3 \).

90% confidence interval will be
\[
10 \pm 1.6449 \times \sqrt{1.3625 \sigma^2} \text{ (i.e. } \pm 1.928) \text{ since } P(Z > 1.6449) = 0.05 \text{ for } Z \sim N(0,1)
\]
Hence constant width = \( 3.842 \), for all \( k \geq 3 \).
(a) Stationary: any roots of $\phi(B)$ must lie outside $\{B \mid |B| = 1\}$ i.e. $|B_i| > 1$.

(b) Invertible: any roots of $\Theta(B)$ outside $\{B \mid |B| = 1\}$.

\[ \phi(B) = 1 - 1.8B + 0.8B^2 = (1 - 0.8B)(1 - B) \]
so roots are $B_1 = \frac{3}{4} = 1.25 > 1$ and $B_2 = 1$ \Rightarrow not stationary.

\[ \Theta(B) = 1 + 0.6B \]
Roots $\approx 0.7$ and $|\frac{1}{0.7}| > 1$ so invertible.

$X_t$ is ARIMA $(1,1,1)$ i.e. $p = 1 = d = q$.

(iii) $a_t = a_t(\theta, \phi) = X_t - 1.8X_{t-1} + 0.8X_{t-2} - 0.6a_{t-1}$

where initially $X_0 = X_{-1} = \mu, X_1 = 0$ and $a_0 = 0$.

$S(\theta, \phi) = \sum \epsilon_t^2(\theta, \phi)$ is then minimised by searching over $\theta$ and $\phi$.

"Conditional" since conditioning on initial choices of $X_0, X_{-1}, a_0$.

(iv) $\phi(B) = 1 - 2.8B + 2.6B^2 - 0.8B^3 = (1 - B)(1 - 1.8B + 0.8B^2)$

$\Theta(B) = 1 - 0.4B - 0.6B^2 = (1 - B)(1 + 0.6B)$

Hence $1 - B$ cancels and the model is redundant.

Simplified model is same as (ii) i.e $(1 - 0.8B)(1 - B)X_t = (1 + 0.6B)A_t$ so ARIMA $(1, 1, 1)$.

(v) If $\phi(B) \mathcal{N}^d X_t = \Theta(B)A_t$ where $\phi(B) = (1 - \omega B)\phi_1(B)$ and $\Theta(B) = (1 - \omega B)\Theta_1(B)$ then

$\epsilon_t(\theta, \phi) = \Theta(B)^{-1}\phi(B)X_t$

$= (1 - \omega B)^{-1}\Theta_1(B)(1 - \omega B)\phi_1(B)X_t$

$= \Theta_1(B)\phi_1(B)X_t$

The errors will not depend on $\omega$. Hence $S(\omega, \phi, \theta)$ will be constant in $\omega$ for any $\phi, \theta$ and the minimisation routine will fail to converge.
Values outside the dotted lines in Figure 2 and 3 are acf or pacf values significantly different from 2 \( \times \sigma (at 5\%) \\
a. White noise rejected as acf has many values outside \\
b. MA(2): acf should drop to \( \pm 2\sigma \) after first two 
   so not a plausible candidate. \\
c. AR(2): pacf drops to \( \pm 2\sigma \) after first two so 
   this is a possibility.

None of the models require differencing:

1. AR(2): 
   \[ X_t = 9.9283 + 0.1158 X_{t-1} + 0.6282 X_{t-2} + \epsilon_t \]

2. AR(3): 
   \[ X_t = 9.9284 + 0.1145 X_{t-1} + 0.6278 X_{t-2} + 0.0037 X_{t-3} + \epsilon_t \]

3. ARMA(2,1): 
   \[ X_t = 9.9285 + 0.1201 X_{t-1} + 0.6272 X_{t-2} + \epsilon_t - 0.0055 \epsilon_{t-1} \]

\( T \)-ratios = Coefficient/SE Judged significantly different (at 5\%) from \( \pm 2\sigma \) if \( |T| > 1.96 \) (as \( n = 157 \) is large)

<table>
<thead>
<tr>
<th>Model</th>
<th>AR(1)</th>
<th>AR(2)</th>
<th>AR(3)</th>
<th>MA(1)</th>
<th>Intercept</th>
</tr>
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<tbody>
<tr>
<td>Model 1</td>
<td>2.07</td>
<td>11.08</td>
<td>-</td>
<td>-</td>
<td>37.23</td>
</tr>
<tr>
<td>Model 2</td>
<td>1.57</td>
<td>10.98</td>
<td>0.05</td>
<td>-</td>
<td>37.07 (AR(1), AR(3), NS)</td>
</tr>
<tr>
<td>Model 3</td>
<td>1.33</td>
<td>10.37</td>
<td>-0.05</td>
<td>-</td>
<td>37.07 (AR(1), MA(1), NS)</td>
</tr>
</tbody>
</table>

Conclusion: AR(2) satisfactory but over-fitting to AR(3) or ARMA(2,1) unnecessary.

The log-likelihoods are identical but AIC is lower for AR(2): more evidence that AR(2) 
should be preferred.

- SD(\( \epsilon_t \)) is higher suggesting that we may 
  have over-differenced.
- acf of \( X_t \) declines exponentially which is 
  characteristic of a stationary series.
- AR(2) provides a satisfactory fit.

Apparent "trend" is probably illusory.