ROYAL STATISTICAL SOCIETY
DATA | EVIDENCE | DECISIONS

THE ROYAL STATISTICAL SOCIETY

2016 EXAMINATIONS − SOLUTIONS

HIGHER CERTIFICATE – MODULE 2

The Society is providing these solutions to assist candidates preparing for the examinations in 2017.

The solutions are intended as learning aids and should not be seen as "model answers".

Users of the solutions should always be aware that in many cases there are valid alternative methods. Also, in the many cases where discussion is called for, there may be other valid points that could be made.

While every care has been taken with the preparation of these solutions, the Society will not be responsible for any errors or omissions.

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RSS HC Module 2 2016 Solutions

Each question has marks out of 20.

Question 1

$$X \sim \text{ Poisson}(\lambda) \qquad x = 0, 1, 2, \ldots$$

$$P_X(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

(i)

$$\begin{aligned} P_X(x+1) &= \frac{\lambda^{x+1} e^{-\lambda}}{(x+1)!} \quad [1] \\ &= \frac{\lambda}{(x+1)} \frac{\lambda^x e^{-\lambda}}{x!} \quad [1] \\ &= \frac{\lambda}{(x+1)} P_x(x) \quad [1] \end{aligned}$$

(ii)

$$\begin{aligned} \text{E}(X) &= \sum_{x=0}^{\infty} x P(X = x) \\ &= \sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} \quad [1] \\ &= \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1} e^{-\lambda}}{(x-1)!} \\ &= \lambda \sum_{y=0}^{\infty} \frac{\lambda^y e^{-\lambda}}{y!} \quad y = x - 1 \quad [1] \\ &= \lambda \times 1 \qquad \text{as sum of density} \\ &= \lambda \quad [1] \end{aligned}$$

$$\begin{aligned} \text{E}(X(X-1)) &= \sum_{x=0}^{\infty} x(x-1) P(X = x) \\ &= \sum_{x=2}^{\infty} x(x-1) \frac{\lambda^x e^{-\lambda}}{x!} \\ &= \lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2} e^{-\lambda}}{(x-2)!} \quad [1] \end{aligned}$$

$$
\begin{aligned}
&= \lambda^2 \sum_{y=0}^{\infty} \frac{\lambda^y e^{-\lambda}}{y!} \quad y = x-2 \\
&= \lambda^2 \times 1 \qquad \text{as sum of density} \\
&= \lambda^2 \quad [1] \\
\mathrm{Var}(X) &= \mathrm{E}(X(X-1)) + \mathrm{E}(X) - (\mathrm{E}(X))^2 \\
&= \lambda^2 + \lambda - \lambda^2 \quad [1] \\
&= \lambda
\end{aligned}
$$

(iii) $Y \sim$ Poisson(μ) independent of X.

$$
\begin{aligned}
P_W(w) &= \sum_{x=0}^{w} P(X=x)P(Y=w-x). \\
&= \sum_{x=0}^{w} \frac{\lambda^x e^{-\lambda}}{x!} \frac{\mu^{w-x} e^{-\mu}}{(w-x)!} \quad [1] \\
&= \sum_{x=0}^{w} \left(\frac{\lambda}{\lambda+\mu}\right)^x \frac{e^{-\lambda}}{x!} \left(\frac{\mu}{\lambda+\mu}\right)^{w-x} \\
&\qquad \times \frac{e^{-\mu}(\lambda+\mu)^w}{(w-x)!} \quad [1] \\
&= \frac{e^{-(\lambda+\mu)}(\lambda+\mu)^w}{w!} \sum_{x=0}^{w} \frac{w!}{x!(w-x)!} \\
&\qquad \times \left(\frac{\lambda}{\lambda+\mu}\right)^x \left(\frac{\mu}{\lambda+\mu}\right)^{w-x} \quad [1] \\
&= \frac{(\lambda+\mu)^w e^{-(\mu+\lambda)}}{w!} \quad [1]
\end{aligned}
$$

$\Rightarrow W \sim$ Poisson($\lambda + \mu$).

(iv) (a) Two conveyor belts - type C and D.
X =number of breakdowns per day on C.
Y =number of breakdowns per day on D.

$$X \sim \text{Poisson}(\lambda = 1.5)$$

$$Y \sim \text{Poisson}(\mu = 0.5)$$

$$
\begin{aligned}
P(\text{exactly one breakdown}) &= P(X=1,Y=0)+P(X=0,Y=1) \\
&= P(X=1)\times P(Y=0)+P(X=0)\times P(Y=1) \quad [1] \\
&= 1.5e^{-1.5}\times e^{-0.5}+e^{-1.5}\times 0.5e^{-0.5} \\
&= e^{-2}(1.5+0.5) \\
&= 2e^{-2} \quad [1] \\
&= 0.2707
\end{aligned}
$$

$$
\begin{aligned}
P(\text{1 breakdown on C}|\text{exactly one breakdown}) &= \frac{P(X=1,Y=0)}{P(\text{exactly one breakdown})} \\
&= \frac{1.5e^{-1.5}\times e^{-0.5}}{2e^{-2}} \quad [1] \\
&= \frac{1.5}{2} \\
&= \frac{3}{4} \quad [1]
\end{aligned}
$$

(b) In 5 days, the rate of occurrence of failures is $5\times 2=10$. Let $T=$ number of failures in 5 days. Then

$$T \sim \text{Poisson}(10)$$

[1,1]

$$P(T\le 1) = 0.0005 \quad [1]$$

Question 2 (i) Let X = number of failures before the first success.

$$
\begin{aligned}
P_X(x) &= (1-p)^x p \quad [1] \\
\text{for} &\quad x=0,1,2,3,4,\ldots \quad [1]
\end{aligned}
$$

$$
\begin{aligned}
\sum_{x=0}^{\infty}(1-p)^x p &= p\sum_{x=0}^{\infty}(1-p)^x \quad [1] \\
&= p\frac{1}{1-(1-p)} \quad \text{sum of GP} \quad [1] \\
&= p\times\frac{1}{p}=1 \quad [1]
\end{aligned}
$$

(ii)

$$\begin{aligned} \mathrm{E}(X) &= \sum_{x=0}^{\infty} x P_X(x) \\ &= \sum_{x=0}^{\infty} x(1-p)^x p \quad [1] \end{aligned}$$

Using the fact that:

$$\begin{aligned} \frac{d}{dp}\sum_{x=0}^{\infty}(1-p)^x &= -\sum_{x=0}^{\infty} x(1-p)^{x-1} \\ &= \frac{d}{dp}\frac{1}{p} \quad [1] \\ &= -\frac{1}{p^2} \quad [1] \end{aligned}$$

Hence

$$\begin{aligned} \mathrm{E}(X) &= p(1-p)\sum_{x=0}^{\infty} x(1-p)^{x-1} \\ &= p(1-p) \times \frac{1}{p^2} \\ &= \frac{(1-p)}{p} \quad [1] \end{aligned}$$

NB Other correct solutions would be given credit.

(iii) (a) When $p = 0.01$,

$$\begin{aligned} P(X = 80) &= 0.99^{79} 0.01 \quad [1] \\ &= 0.00452 \quad [1] \end{aligned}$$

(b) EITHER

$$\begin{aligned} P(X > 80) &= 1 - P(X \leq 80) \quad [1] \\ &= 1 - \sum_{x=0}^{80} x(1-p)^x p \quad [1] \\ &= 1 - p\left(\frac{1-(1-p)^{80}}{1-(1-p)}\right) \quad [1] \\ &= 1 - 0.01(\frac{(1-0.99^{80}))}{0.01} \quad [1] \\ &= 1 - 0.5525 \\ &= 0.4475 \quad [1] \end{aligned}$$

OR As it will take more than 80 attempts if and only if the first 80 are all failures **[2]**
(Either get 2 marks or 0)

$$\begin{aligned} P(X > 80) &= 0.99^{80} \quad [2] \\ & \qquad\qquad \text{(either get 2 marks or 0)} \\ &= 0.4475. \quad [1] \end{aligned}$$

(iv) Let Y= total number of failures before the second success. The probability mass function is

$$P_Y(y) = \binom{y+1}{1}(1-p)^y p^2 = (y+1)(1-p)^y p^2 \qquad y = 0, 1, 2, \ldots \quad [1]$$

The experiment to obtain Y can be considered as running the first experiment until the first success and then repeating an independent experiment to obtain a further success. **[1]**
Hence

$$\begin{aligned} \mathrm{E}(Y) &= \mathrm{E}(X_1) + \mathrm{E}(X_2) \quad [1] \\ &= \frac{(1-p)}{p} + \frac{(1-p)}{p} \quad [1] \\ &= 2\frac{(1-p)}{p} \end{aligned}$$

Question 3 (a) Let L be the volume of the large bottle in litres.
Let S be the volume of the small bottle in litres.

$$L \sim N(1.5, 0.01^2)$$
$$S \sim N(0.5, 0.008^2)$$

(i) Take a random sample of 1 large bottle and 3 small bottles.

$$P(L > 3S) = P(L - 3S > 0) \quad [1]$$

Let $Y = L - 3S$.

$$\mathrm{E}(L - 3S) = \mathrm{E}(L) - 3\,\mathrm{E}(S) = 1.5 - 3 \times 0.5 = 0$$

[1]

$$\mathrm{Var}(L - 3S) = \mathrm{Var}(L) + 9 * \mathrm{Var}(S) = 0.01^2 + 9 * 0.008^2 = 0.000676$$

$$sd(L - 3S) = 0.026$$

NB The variance is not needed here, but some may not realise that. It is not necessary to calculate, or even mention, the variance.

$$\begin{aligned} P(Y > 0) &= P\left(Z > \frac{0 - 0}{0.026}\right) \\ &= P(Z > 0) \\ &= 1 - 0.5 \\ &= 0.5 \quad [1] \end{aligned}$$

(ii) Let $T = 10L + 3S$.

$$\begin{aligned} \mathrm{E}(10L + 3S) &= 10\,\mathrm{E}(L) + 3\,\mathrm{E}(S) \quad [1] \\ &= 10 \times 1.5 + 3 \times 0.5 \\ &= 16.5 \quad [1] \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(10L + 3S) &= 10^2\,\mathrm{Var}(L) + 3^2\,\mathrm{Var}(S) \quad [1] \\ &= 100 \times 0.01^2 + 9 \times 0.008^2 \\ &= 0.01 + 0.000576 \\ &= 0.010576 \quad [1] \end{aligned}$$

T is a linear combination of normal r.v.'s so T is also normal **[1]**

$$T \sim N(16.5, 0.010576) \quad [1]$$

(b) $n = 50$, $\mu = 30$g and $\sigma = 5$g.

(i) As n is large the distribution of the sample mean tends to normality by the Central Limit Theorem. **[1]**

$$\mathrm{E}(\bar{X}) = \mu = 30\text{g}$$

[1]

$$\mathrm{Var}(\bar{X}) = \frac{\sigma^2}{n} = \frac{25}{50} = \frac{1}{2}$$

[1]

$$\bar{X} \sim N(30, 0.5)$$

(ii)

$$\begin{aligned} P(\bar{X} < 29) &= P\left(Z < \frac{29-30}{\sqrt{0.5}}\right) \quad [1] \\ &= P(Z < -\sqrt{2}) \\ &= P(Z < -1.41) \quad [1] \\ &= P(Z > 1.41) \\ &= 1 - P(Z < 1.41) \\ &= 1 - 0.9207 \\ &= 0.0793 \quad [1] \end{aligned}$$

(iii) Sample of size n.

$$P\left(Z < \frac{29-30}{\frac{5}{\sqrt{n}}}\right) = 0.05 \quad [1]$$

$\Rightarrow$

$$\begin{aligned} \frac{29-30}{\frac{5}{\sqrt{n}}} &< -1.645 \quad [1] \\ -1 &< -1.645 \times \frac{5}{\sqrt{n}}` \quad [1] \\ -\sqrt{n} &< -1.634 \times 5 \\ \sqrt{n} &> 8.225 \quad [1] \end{aligned}$$

The minimum value of n is 68. **[1]**

Question 4 (a) The continuous random variable X is distributed with probability density function

$$f(x) = \alpha(1-x)^{\alpha-1} \qquad 0 < x < 1; \alpha > 0$$

(i)

$$\begin{aligned} F(x) &= \int_0^x \alpha(1-t)^{\alpha-1} dt \quad [1] \\ &= \alpha\left[-\frac{(1-x)^{\alpha}}{\alpha}\right]_0^x \\ &= -(1-x)^{\alpha} + 1 \\ &= 1 - (1-x)^{\alpha} \quad [1] \end{aligned}$$

$$
\begin{aligned}
F(x) &= 0 \qquad x \leq 0\,[0.5] \\
&= 1-(1-x)^{\alpha} \\
&= 1 \qquad x \geq 1\,[0.5]
\end{aligned}
$$

(ii)

$$
\begin{aligned}
P(0.25 < X < 0.75) &= F(0.75) - F(0.25) \quad [1] \\
&= 1-(1-0.75)^{\alpha} - [1-(1-0.25)^{\alpha}] \quad [1] \\
&= 0.75^{\alpha} - 0.25^{\alpha} \quad [1] \\
&= 0.25^{\alpha}(3^{\alpha}-1) \\
&= \frac{3^{\alpha}-1}{4^{\alpha}} \quad [1]
\end{aligned}
$$

(iii) Let M be the median.

$$F(M) = \frac{1}{2} \quad [1]$$

$$
\begin{aligned}
1-(1-M)^{\alpha} &= \frac{1}{2} \\
(1-M)^{\alpha} &= \frac{1}{2} \quad [1] \\
1-M &= 2^{-\frac{1}{\alpha}} \\
M &= 1-2^{-\frac{1}{\alpha}} \quad [1]
\end{aligned}
$$

(b)

$$Y \sim Ga(2,3) \quad f(y) = 9ye^{-3y}$$

(i)

$$
\begin{aligned}
\mathrm{E}(Y) &= \int_0^{\infty} yf(y)dy \\
&= 9\int_0^{\infty} y^2 e^{-3y}dy \quad [1] \\
&= 9\left[\left[\frac{y^2e^{-3y}}{(-3)}\right]_0^{\infty} + \frac{2}{3}\int_0^{\infty} ye^{-3y}dy\right] \quad [1] \\
&= 6\int_0^{\infty} ye^{-3y}dy \quad [1] \\
&= 6\left[\left[\frac{ye^{-3y}}{(-3)}\right]_0^{\infty} + \frac{1}{3}\int_0^{\infty} e^{-3y}dy\right] \quad [1]
\end{aligned}
$$

$$
\begin{aligned}
&= 2\int_0^\infty e^{-3y}dy \\
&= 6\left[\frac{e^{-3y}}{(-3)}\right]_0^\infty \quad [1] \\
&= \frac{2}{3} \quad [1]
\end{aligned}
$$

NB in line 3 the last integral is a multiple of the integral of the pdf and hence integrates to 1, hence the last term is just $\frac{2}{3} \times 1$. Full marks are to be given for this solution.

(ii)

$$
\begin{aligned}
P(X < 3) &= 9\int_0^3 xe^{-3x}dx \quad [1] \\
&= 9\left[\left[\frac{xe^{-3x}}{(-3)}\right]_0^3 - \int_0^3 \frac{e^{-3x}}{(-3)}dx\right] \quad [1] \\
&= -9e^{-9} + 3\int_0^3 e^{-3x}dx \\
&= -9e^{-9} + 3\left[\frac{e^{-3x}}{(-3)}\right]_0^3 \quad [1] \\
&= 1 - 10e^{-9} \quad [1] \\
&= 0.9988
\end{aligned}
$$

NB The final evaluation is not required for the final mark.