The Society is providing these solutions to assist candidates preparing for the examinations in 2017.

The solutions are intended as learning aids and should not be seen as "model answers".

Users of the solutions should always be aware that in many cases there are valid alternative methods. Also, in the many cases where discussion is called for, there may be other valid points that could be made.

While every care has been taken with the preparation of these solutions, the Society will not be responsible for any errors or omissions.

The Society will not enter into any correspondence in respect of these solutions.
1. (a) (i)  $P(A|B) = \frac{P(A \cap B)}{P(B)}$  (0 for numerator, 1 for denominator)

(ii) (A) If $A$ and $B$ are independent, $P(A \cap B) = P(A) \times P(B)$

$\Rightarrow P(A|B) = \frac{P(A)P(B)}{P(B)} = P(A) \quad \text{[1]}$

(b) $A$ and $B$ are mutually exclusive if $P(A \cap B) = 0$

$\Rightarrow P(A|B) = 0 \quad \text{[1]}$

(c) $B \subset A \Rightarrow P(A \cap B) = P(B)$

$\Rightarrow P(A|B) = \frac{P(B)}{P(B)} = 1 \quad \text{[1]}

(b) Let $D$ be the person has disease
- + be positive test result
- - be negative test result

$P(+) = 0.98$  $P(-) = 0.99$

$P(D) = 0.03$

(i) Want $P(D|+)$

$= \frac{P(+|D)P(D)}{P(+)}$

$P(+) = P(+)P(D) + P(+)P(\overline{D})P(\overline{D}) \quad \text{[1]}

= 0.98 \times 0.03 + 0.01 \times 0.97

= 0.0385 \quad \text{[1]}

(ii) $P(\overline{D}|+) = \frac{0.98 \times 0.03}{0.0385} = 0.7634 \quad \text{[1]}

$P(D|+) = \frac{P(+|D)P(D)}{P(+)P(D) + P(+)P(\overline{D})P(\overline{D})} \quad \text{[0]}

= \frac{0.98^2 \times 0.03}{0.98 \times 0.03 + 0.01 \times 0.97} \quad \text{[0] = 0.9966} \quad \text{[1]}
(c) \( P(M \text{ wins}) = \frac{1}{2} \hspace{1cm} P(M \text{ draws}) = \frac{1}{6} \hspace{1cm} P(M \text{ loses}) = \frac{1}{3} \)
\( P(C \text{ wins}) = \frac{2}{3} \hspace{1cm} P(C \text{ draws}) = \frac{1}{6} \hspace{1cm} P(C \text{ loses}) = \frac{1}{6} \)

(i) \( M \text{ wins if } M \text{ wins and } C \text{ does not win or } M \text{ draws and } C \text{ loses} \)

\[ P(M \text{ wins championship}) = P(M \text{ wins } \& C \text{ does not win}) + P(M \text{ draws } \& C \text{ loses}) \]
\[ = P(M \text{ wins}) \times P(C \text{ does not win}) + P(M \text{ draws}) \times P(C \text{ loses}) \]
\[ = \frac{1}{2} \times \frac{1}{3} + \frac{1}{6} \times \frac{1}{6} \]
\[ = \frac{7}{36} \]

(ii) \( P(C \text{ draws } | M \text{ wins championship}) \)

\[ = \frac{P(C \text{ draws } \& M \text{ wins championship})}{P(M \text{ wins championship})} \]
\[ = \frac{P(C \text{ draws}) \times P(M \text{ wins game})}{P(M \text{ wins championship})} \]
\[ = \frac{\frac{1}{6} \times \frac{1}{2}}{\frac{1}{36}} \]
\[ = \frac{2}{7} \]
2. (a) A continuity correction is needed when a discrete random variable is approximated by a continuous random variable.

\[ P(X \leq x) = P(X < x + 1) \]

and so you need to allow for this in your approximation.

(b) Poisson approximation to Binomial is used when \( n \) is large and \( \mu/n \) small (or \( n \) large and \( \mu/n \) large) interchange success failure.

Possible values \( p < 0.1 \) or \( p > 0.9 \) and \( \mu n > 5 \).

Normal approximation to Binomial is used when \( n \) is large and \( \mu/n \) is not small, i.e. \( 0.1 < p < 0.9 \); better approximation if \( p \) closer to 0.5.

(c) Fair die \( n = 300 \quad P(X < 45 \text{ sides}) \)

Let \( X = \text{no. of sides} \quad X \sim \text{Binomial} (300, \frac{1}{6}) \)

Use a normal approximation to Binomial with \( \mu = 50, \sigma^2 = 125 \)

\[ P(X < 45) = P(X \leq 44) \]

\[ = P \left( Z < \frac{44.5 - 50}{\sqrt{125/3}} \right) \]

\[ = P \left( Z < -0.852 \right) \]

\[ = 1 - P \left( Z > 0.852 \right) \]

\[ = 1 - \Phi (0.852) \]

\[ = 1 - 0.80298 \]

\[ = 0.19702 \]

No marks if use a Poisson approximation.

(d) \( X = \text{no. of accidents in one working week} \quad X \sim \text{Poisson}(0.2) \)

Let \( Y = \text{no. of accidents in 150 working weeks} \quad Y \sim \text{Poisson} (30 \cdot 0) \)
2. (d) continued.

\[ P (Y \geq 35) \quad \mu = 30.0 \quad \sigma^2 = 30.0 \]

Use a Normal approximation to Poisson.

\[ P (Y \geq 35) = P \left( Z > \frac{34.5 - 30.0}{\sqrt{30.0}} \right) \]

\[ = P (Z > 0.8216) \]

\[ = 1 - P (Z < 0.8216) \]

\[ = 1 - 0.7943 \quad \text{using interpolation} \]

\[ = 0.2057 \]
3. \( f(y) = \begin{cases} \frac{1}{2} (y+y^3) & 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases} \)

(i) \( \int_0^2 f(y) \, dy = 1 \) \hfill (1)

\[ \frac{1}{2} \int_0^2 (y+y^3) \, dy = 1 \] \hfill (1)

\[ \frac{1}{2} \left[ \frac{y^2}{2} + \frac{y^4}{4} \right]_0^2 = 1 \] \hfill (1)

\[ \frac{1}{2} [2+4] = 1 \Rightarrow k = \frac{1}{6} \] \hfill (1)

(ii) \( F(y) = \mathbb{P}(Y \leq y) \) \hfill (1)

\[ = \int_0^y \frac{1}{6} (u+u^3) \, du \] \hfill (1)

\[ = \left[ \frac{1}{6} \left( \frac{u^2}{2} + \frac{u^4}{4} \right) \right]_0^y \] \hfill (1)

\[ = \frac{1}{6} \left( \frac{y^2}{2} + \frac{y^4}{4} \right) \] \hfill (1)

\[ = \frac{y^2}{12} \left( \frac{y^2+2}{2} \right) \quad 0 \leq y \leq 2 \] \hfill (1)

\[ \therefore F(y) = \begin{cases} \frac{y^2}{12} \left( \frac{y^2+2}{2} \right) & 0 \leq y \leq 2 \\ 1 & y \geq 2 \end{cases} \]

\[ P \left( \frac{1}{2} < y < \frac{3}{2} \right) = F \left( \frac{3}{2} \right) - F \left( \frac{1}{2} \right) \] \hfill (1)

\[ = \frac{\left( \frac{3}{2} \right)^2 \left( \frac{3}{2} \right)^2 + 2}{12} - \frac{\left( \frac{1}{2} \right)^2 \left( \frac{1}{2} \right)^2 + 2}{12} \] \hfill (1)

\[ = \frac{9}{48} \left( \frac{9+8}{8} \right) - \frac{1}{48} \left( \frac{1+8}{8} \right) = \frac{3}{8} \] \hfill (1)
(iii) \( \text{Var} \{ Y \} = E[Y^2] - (E[Y])^2 \)  

\[
E[Y] = \frac{1}{6} \int_0^2 y \,(y + y^3) \,dy \quad \circled{1}
\]

\[
= \frac{1}{6} \left[ \frac{y^3}{3} + \frac{y^5}{5} \right]_0^1 \quad \circled{1}
\]

\[
= \frac{1}{6} \left[ \frac{8}{3} + \frac{32}{5} \right] \quad \circled{1}
\]

\[
= \frac{1}{6} \left[ \frac{40 + 96}{15} \right] = \frac{136}{90} = 1.511 \left( \frac{68}{45} \right) \quad \circled{1}
\]

\[
E[Y^2] = \frac{1}{6} \int_0^2 y^2 \,(y + y^3) \,dy \quad \circled{1}
\]

\[
= \frac{1}{6} \left[ \frac{y^4}{4} + \frac{y^6}{6} \right]_0^1 \quad \circled{1}
\]

\[
= \frac{1}{6} \left[ \frac{4 + 32}{3} \right] = \frac{44}{18} = \frac{22}{9} \quad \circled{1}
\]

\[
\therefore \text{Var} \{ Y \} = E[Y^2] - (E[Y])^2
\]

\[
= \frac{22}{9} - \left( \frac{68}{45} \right)^2 = 0.161 \quad \circled{1}
\]
4. \( X_1, X_2, \ldots, X_n \) are independent \( X_i \sim N(\mu_i, \sigma^2_i) \) \( i = 1, 2, \ldots, n \)

\[ T = \sum_{i=1}^{n} X_i \]

(i) \( a \) \( \mathbb{E}[T] = \sum_{i=1}^{n} \mu_i \) \[ 1 \]

(b) \( \text{Var}[T] = \sum_{i=1}^{n} \sigma^2_i \) \[ 1 \] by independence

(c) Normal \[ 1 \]

(ii) Let \( X_i \) be weight of each hardback book \( i = 1, 2, 3 \)

\( X_i \sim N(1.5, 0.5^2) \)

Let \( Y_j \) be weight of each paperback book \( j = 1, 2, \ldots, 6 \)

\( Y_j \sim N(0.8, 0.3^2) \)

(iii) Let \( W_i \) be total weight of parcel \( \Rightarrow W = X_1 + X_2 + X_3 + Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6 \)

\( \mathbb{E}[W_i] = \sum_{i=1}^{3} \mathbb{E}[X_i] + \sum_{j=1}^{6} \mathbb{E}[Y_j] \)

\[ = 3 \times 1.5 + 6 \times 0.8 = 9.3 \] \[ 1 \]

\( \text{Var}[W_i] = \sum_{i=1}^{3} \text{Var}[X_i] + \sum_{j=1}^{6} \text{Var}[Y_j] \) \[ 1 \] by independence

\[ = 3 \times 0.5^2 + 6 \times 0.3^2 = 1.29 \]

\( P(W_i > 11) = P\left( \frac{Z > 11 - 9.3}{\sqrt{1.29}} \right) \)

\( = P(Z > 1.4968) \)

\( = 1 - \Phi(1.4968) \) \[ 1 \]

\( = 1 - 0.9328 \)

\( = 0.0672 \) \[ 1 \]
(b) \[ P(X_1 + X_2 + X_3 > Y_1 + Y_2 + \ldots + Y_6) \]

\[ = P(X_1 + X_2 + X_3 - Y_1 - Y_2 - \ldots - Y_6 > 0) \]

Let \( W_2 = X_1 + X_2 + X_3 - Y_1 - Y_2 - Y_3 - Y_4 - Y_5 - Y_6 \)

\[
E[W_2] = \sum_{i=1}^{3} E[X_i] - \sum_{i=1}^{6} E[Y_i] = 3 \times 1.5 - 6 \times 0.8 = -0.3 \tag{1}
\]

\[
\text{Var}[W_2] = \sum_{i=1}^{3} \text{Var}[X_i] + \sum_{i=1}^{6} \text{Var}[Y_i] \quad \text{by indy} \]

\[ = 1.29 \tag{1} \]

\[
P(W_2 > 0) = P\left( Z > \frac{0 - (-0.3)}{\sqrt{1.29}} \right) \tag{1}
\]

\[ = P(Z > 0.2641) \]

\[ = 1 - \Phi(0.2641) \tag{1} \]

\[ = 1 - 0.6042 \]

\[ = 0.3958 \tag{1} \]

(c) Let \( n_H \) be the maximum number of hardback books on pallet

Let \( W_3 = \sum_{i=1}^{n_H} X_i \)

\[ E[W_3] = 1.5n_H \tag{1} \]

\[ \text{Var}[W_3] = 0.5^2n_H = 0.25n_H \tag{1} \]

\[ P(W_3 > 100) < 0.01 \Rightarrow P\left( Z > \frac{100 - 1.5n_H}{\sqrt{0.25n_H}} \right) < 0.01 \]

\[ \Rightarrow 100 - 1.5n_H > 2.3263 \Rightarrow 100 - 1.5n_H > 1.1632\sqrt{n_H} \tag{1} \]

\[ \Rightarrow 1.5n_H + 1.1632\sqrt{n_H} - 100 < 0 \tag{1} \]